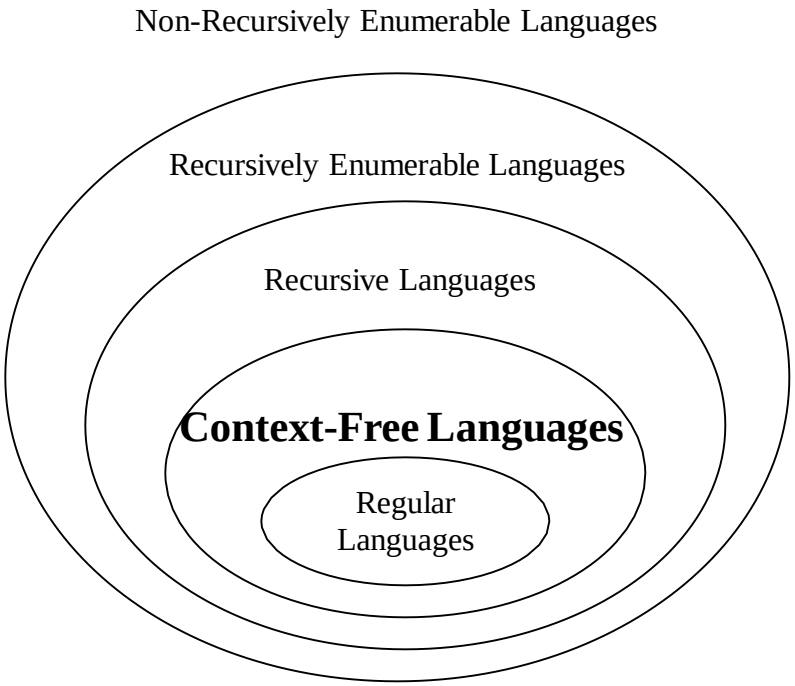
Subject: Theory of computation
Topic: PDA
Name of the teacher: Lisna Thomas
Academic year:2020-21

Hierarchy of languages

Regular Languages Finite State Machines, Regular Expression

Context Free Languages Context Free Grammar, **Push-down Automata**



Pushdown Automata (PDA)

- **Informally:**
  - A PDA is an NFA-ε with a stack.
  - Transitions are modified to accommodate stack operations.

- **Questions:**
  - What is a stack?
  - How does a stack help?

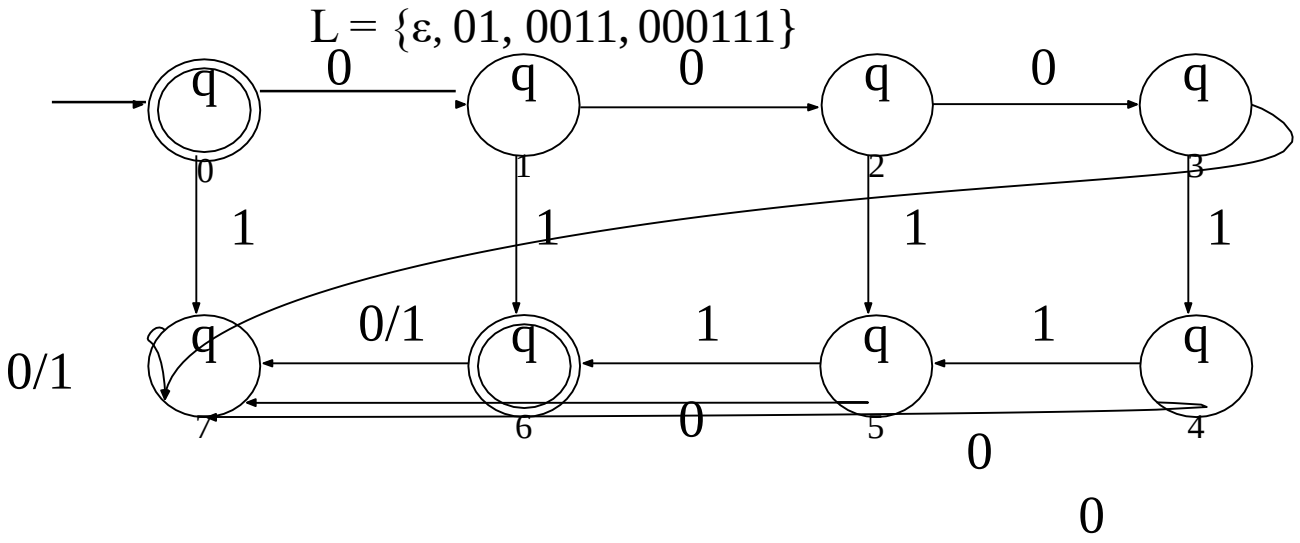
- A DFA can "remember" only a finite amount of information, whereas a PDA can "remember" an infinite amount of (certain types of) information, in one memory-stack

- **Example:**

  $\{0^n1^n \mid 0=<n\}$ is *not* regular, but

  $\{0^n1^n \mid 0\leq n\leq k$, for some fixed k$\}$ is regular, for any fixed k.

- **For k=3:**



- In a DFA, each state remembers a finite amount of information.
- To get $\{0^n1^n \mid 0 \leq n\}$ with a DFA would require an infinite number of states using the preceding technique.
- An infinite stack solves the problem for $\{0^n1^n \mid 0 \leq n\}$ as follows:
  - Read all 0's and place them on a stack
  - Read all 1's and match with the corresponding 0's on the stack
- Only need two states to do this in a PDA
- Similarly for $\{0^n1^m0^{n+m} \mid n,m \geq 0\}$

Formal Definition of a PDA

- A pushdown automaton (PDA) is a seven-tuple:

$M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$

Q	A finite set of states
Σ	A finite input alphabet
Γ	A finite stack alphabet
q_0	The initial/starting state, q_0 is in Q
z_0	A starting stack symbol, is in Γ // need not always remain at the bottom of stack
F	A set of final/accepting states, which is a subset of Q
δ	A transition function, where

$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow$ finite subsets of $Q \times \Gamma^*$

- Consider the various parts of δ:

  $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow$ finite subsets of $Q \times \Gamma^*$

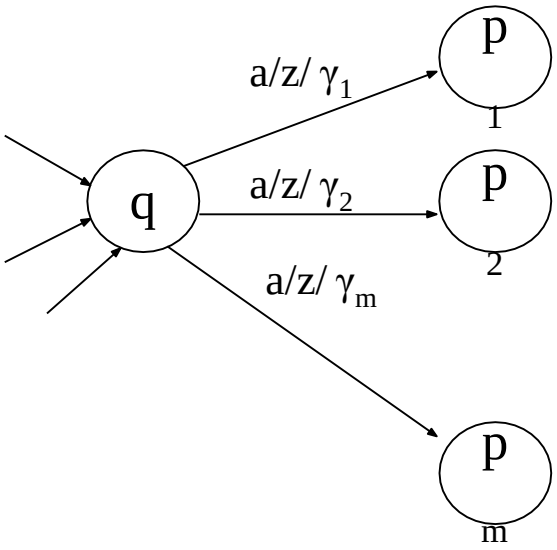
  - Q on the LHS means that at each step in a computation, a PDA must consider its' current state.
  - Γ on the LHS means that at each step in a computation, a PDA must consider the symbol on top of its' stack.
  - $\Sigma \cup \{\varepsilon\}$ on the LHS means that at each step in a computation, a PDA may or may not consider the current input symbol, i.e., it may have epsilon transitions.

  - "Finite subsets" on the RHS means that at each step in a computation, a PDA may have several options.
  - Q on the RHS means that each option specifies a new state.
  - Γ^* on the RHS means that each option specifies zero or more stack symbols that will replace the top stack symbol, but *in a specific sequence*.

- **Two types of PDA transitions:**

$\delta(q, a, z) = \{(p_1,\gamma_1), (p_2,\gamma_2),\ldots, (p_m,\gamma_m)\}$

  - Current state is q
  - Current input symbol is a
  - Symbol currently on top of the stack z
  - Move to state p_i from q
  - Replace z with γ_i on the stack (leftmost symbol on top)
  - Move the input head to the next input symbol



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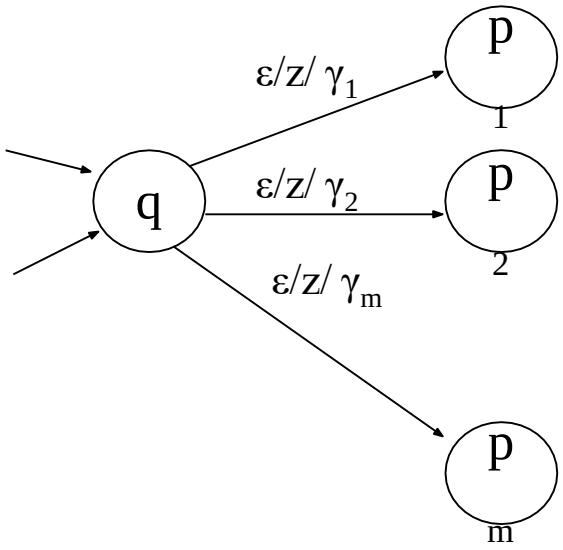
- **Two types of PDA transitions:**

$\delta(q, \varepsilon, z) = \{(p_1,\gamma_1), (p_2,\gamma_2),\ldots, (p_m,\gamma_m)\}$

  - Current state is q
  - Current input symbol is not considered
  - Symbol currently on top of the stack z
  - Move to state p_i from q
  - Replace z with γ_i on the stack (leftmost symbol on top)
  - **No input symbol is read**



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- **Example:** 0^n1^n, n>=0

  $M = (\{q_1, q_2\}, \{0, 1\}, \{L, \#\}, \delta, q_1, \#, \emptyset)$

  δ:

  (1) $\delta(q_1, 0, \#) = \{(q_1, L\#)\}$ // stack order: L on top, then # below

  (2) $\delta(q_1, 1, \#) = \emptyset$ // illegal, string rejected, *When will it happen?*

  (3) $\delta(q_1, 0, L) = \{(q_1, LL)\}$

  (4) $\delta(q_1, 1, L) = \{(q_2, \varepsilon)\}$

  (5) $\delta(q_2, 1, L) = \{(q_2, \varepsilon)\}$

  (6) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$ //if ε read & stack hits bottom, accept

  (7) $\delta(q_2, \varepsilon, L) = \emptyset$ // illegal, string rejected

  (8) $\delta(q_1, \varepsilon, \#) = \{(q_2, \varepsilon)\}$ // n=0, accept

- **Goal:** (acceptance)
  - Read the entire input string
  - Terminate with an empty stack

- Informally, a string is accepted if there exists a computation that uses up all the input and leaves the stack empty.
- *How many rules should be there in delta?*

- **Language: 0^n1^n, n>=0**

δ:

(1) **$\delta(q_1, 0, \#) = \{(q_1, L\#)\}$ // stack order: L on top, then # below**

(2) **$\delta(q_1, 1, \#) = \emptyset$ // illegal, string rejected, *When will it happen?***

(3) **$\delta(q_1, 0, L) = \{(q_1, LL)\}$**

(4) **$\delta(q_1, 1, L) = \{(q_2, \varepsilon)\}$**

(5) **$\delta(q_2, 1, L) = \{(q_2, \varepsilon)\}$**

(6) **$\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$ //if ε read & stack hits bottom, accept**

(7) **$\delta(q_2, \varepsilon, L) = \emptyset$ // illegal, string rejected**

(8) **$\delta(q_1, \varepsilon, \#) = \{(q_2, \varepsilon)\}$ // n=0, accept**

- **0011**
- ***(q1, 0 011, #) |-***
  (q1, 0 11, L#) |-
  (q1, 1 1, LL#) |-
  (q2, 1, L#) |-
  (q2, e, #) |-
  (q2, e, e): accept
- ***011***
- ***(q1, 0 11, #) |-***
  (q1, 1 1, L#) |-
  (q2, 1, #) |-
  Ø : reject
- ***Try 001***

- **Example:** balanced parentheses,
- e.g. in-language: ((())()), or (())(), but not-in-language: ((())

M = ({q_1}, {"(", ")"}, {L, #}, δ, q_1, #, Ø)

δ:

(1) δ(q_1, (, #) = {(q_1, L#)} // stack order: L-on top-then- # lower

(2) δ(q_1,), #) = Ø // illegal, string rejected

(3) δ(q_1, (, L) = {(q_1, LL)}

(4) δ(q_1,), L) = {(q_1, ε)}

(5) δ(q_1, ε, #) = {(q_1, ε)} //if ε read & stack hits bottom, accept

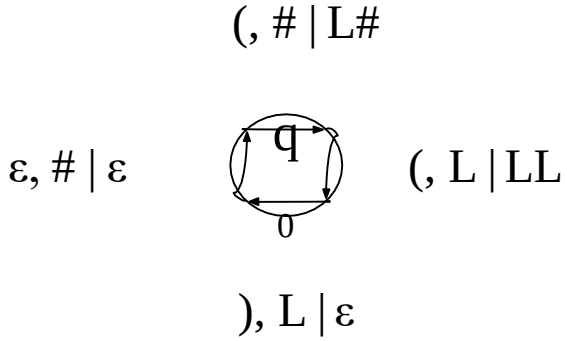
(6) δ(q_1, ε, L) = Ø // illegal, string rejected

// *What does it mean? When will it happen?*

- **Goal:** (acceptance)
  - Read the entire input string
  - Terminate with an empty stack

- Informally, a string is accepted if there exists a computation that uses up all the input and leaves the stack empty.
- *How many rules should be in delta?*

- Transition Diagram:



- Example Computation:

Current Input	Stack	Transition	
(())	#		-- initial status
())	L#		(1)
- Could have applied rule (5), but			
))	LL#		(3)
it would have done no good			
)	L#		(4)
ε	#		(4)
ε	-		(5)

- **Example PDA #1:** For the language {x | x = wcw^r and *w* in {0,1}*, butsigma={0,1,c}}
- *Is this a regular language?*
- *Note: length |x| is odd*

M = ({q_1, q_2}, {0, 1, c}, {#, B, G}, δ, q_1, #, Ø)

δ:

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 0, B) = \{(q_1, BB)\}$

(3) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(4) $\delta(q_1, c, \#) = \{(q_2, \#)\}$

(5) $\delta(q_1, c, B) = \{(q_2, B)\}$

(6) $\delta(q_1, c, G) = \{(q_2, G)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(10) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(11) $\delta(q_1, 1, G) = \{(q_1, GG)\}$

(12) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

- **Notes:**
  - Stack grows leftwards
  - Only rule #8 is non-deterministic.
  - Rule #8 is used to pop the final stack symbol off at the end of a computation.

- **Example Computation:**

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 0, B) = \{(q_1, BB)\}$

(3) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(4) $\delta(q_1, c, \#) = \{(q_2, \#)\}$

(5) $\delta(q_1, c, B) = \{(q_2, B)\}$

(6) $\delta(q_1, c, G) = \{(q_2, G)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(10) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(11) $\delta(q_1, 1, G) = \{(q_1, GG)\}$

(12) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied	Rules Applicable
q_1	**0**1c10	#		
			(1)	
q_1	**1**c10	B#		(1)
			(10)	
q_1	**c**10	GB#		(10)
			(6)	
q_2	**1**0	GB#		(6)

- **Example Computation:**

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 0, B) = \{(q_1, BB)\}$

(3) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(4) $\delta(q_1, c, \#) = \{(q_2, \#)\}$

(5) $\delta(q_1, c, B) = \{(q_2, B)\}$

(6) $\delta(q_1, c, G) = \{(q_2, G)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(10) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(11) $\delta(q_1, 1, G) = \{(q_1, GG)\}$

(12) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied
q_1	**1**c1	#	
q_1	**c**1	G#	(9)
q_2	**1**	G#	(6)
q_2	**ε**	#	(12)
q_2	ε	ε	(8)

- **Questions:**
  - Why isn't $\delta(q_2, 0, G)$ defined?
  - Why isn't $\delta(q_2, 1, B)$ defined?
- TRY: 11c1

- **Example PDA #2:** For the language $\{x \mid x = ww^r \text{ and } w \text{ in } \{0,1\}^*\}$
- *Note: length |x| is even*

$M = (\{q_1, q_2\}, \{0, 1\}, \{\#, B, G\}, \delta, q_1, \#, \emptyset)$

δ:

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(3) $\delta(q_1, 0, B) = \{(q_1, BB), (q_2, \varepsilon)\}$

(4) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(5) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(6) $\delta(q_1, 1, G) = \{(q_1, GG), (q_2, \varepsilon)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, \varepsilon, \#) = \{(q_2, \#)\}$

(10) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

- **Notes:**
  - Rules #3 and #6 are non-deterministic: two options each
  - Rules #9 and #10 are used to pop the final stack symbol off at the end of a computation.

- **Example Computation:**

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(3) $\delta(q_1, 0, B) = \{(q_1, BB), (q_2, \varepsilon)\}$

(4) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(5) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(6) $\delta(q_1, 1, G) = \{(q_1, GG), (q_2, \varepsilon)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

(10) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied	Rules Applicable
q_1	000000	#		(1), (9)
q_1	00000	B#	(1)	(3), both options
q_1	0000	BB#	(3) option #1	(3), both options
q_1	000	BBB#	(3) option #1	(3), both options
q_2	00	BB#	(3) option #2	(7)
q_2	0	B#	(7)	(7)
q_2	ε	#	(7)	(10)
q_2	ε	ε	(10)	

- **Negative Example Computation:**

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(3) $\delta(q_1, 0, B) = \{(q_1, BB), (q_2, \varepsilon)\}$

(4) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(5) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(6) $\delta(q_1, 1, G) = \{(q_1, GG), (q_2, \varepsilon)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

(10) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied
q_1	000	#	
q_1	00	B#	(1)
q_1	0	BB#	(3) option #1 (q2, 0, #) by option 2
q_1	ε	BBB#	(3) option #1 -crashes, no-rule to apply-
			(q2, ε, B#) by option 2
			-rejects: end of string but not empty stack-

- **Example Computation:**

(1) $\delta(q_1, 0, \#) = \{(q_1, B\#)\}$

(2) $\delta(q_1, 1, \#) = \{(q_1, G\#)\}$

(3) $\delta(q_1, 0, B) = \{(q_1, BB), (q_2, \varepsilon)\}$

(4) $\delta(q_1, 0, G) = \{(q_1, BG)\}$

(5) $\delta(q_1, 1, B) = \{(q_1, GB)\}$

(6) $\delta(q_1, 1, G) = \{(q_1, GG), (q_2, \varepsilon)\}$

(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$

(8) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

(9) $\delta(q_1, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

(10) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied
q_1	010010	#	
q_1	10010	B#	(1)
From (1) and (9)			
q_1	0010	GB#	(5)
q_1	010	BGB#	(4)
q_2	10	GB#	(3) option #2
q_2	0	B#	(8)
q_2	ε	#	(7)
q_2	ε	ε	(10)

- **Exercises:**
  - 0011001100 // *how many total options the machine (or you!) may need to try before rejection?*
  - 011110
  - 0111

Formal Definitions for PDAs

- Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ be a PDA.

- **Definition:** An *instantaneous description* (ID) is a triple (q, w, γ), where q is in Q, w is in Σ^* and γ is in Γ^*.
  - q is the current state
  - w is the unused input
  - γ is the current stack contents

- **Example:** (for PDA#2)

$(q_1, 111, GBR)$ $(q_1, 11, GGBR)$

$(q_1, 111, GBR)$ $(q_2, 11, BR)$

$(q_1, 000, GR)$ $(q_2, 00, R)$

- Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ be a PDA.

- **Definition:** Let a be in $\Sigma \cup \{\varepsilon\}$, w be in Σ^*, z be in Γ, and α and β both be in Γ^*. Then:

$$(q, aw, z\alpha) \vdash_M (p, w, \beta\alpha)$$

  if $\delta(q, a, z)$ contains (p, β).

- Intuitively, if I and J are instantaneous descriptions, then I $\vdash$ J means that J follows from I by one transition.

- **Examples:** (PDA #2)

  $(q_1, 111, GBR) \vdash (q_1, 11, GGBR)$ (6) option #1, with a=1, z=G, β=GG, w=11, and α= BR

  $(q_1, 111, GBR) \vdash (q_2, 11, BR)$ (6) option #2, with a=1, z=G, β= ε, w=11, and α= BR

  $(q_1, 000, GR) \vdash (q_2, 00, R)$ Is *not* true, For any a, z, β, w and α

- **Examples:** (PDA #1)

  $(q_1, (())), L\#) \vdash (q_1, ())),LL\#)$ (3)

- **Definition:** |—* is the reflexive and transitive closure of |—.
  - I |—* I for each instantaneous description I
  - If I |— J and J |—* K then I |—* K

- Intuitively, if I and J are instantaneous descriptions, then I |—* J means that J follows from I by zero or more transitions.

- **Definition:** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ be a PDA. The *language accepted by empty stack*, denoted $L_E(M)$, is the set

$$\{w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \varepsilon) \text{ for some } p \text{ in } Q\}$$

- **Definition:** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ be a PDA. The *language accepted by final state*, denoted $L_F(M)$, is the set

$$\{w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \gamma) \text{ for some } p \text{ in } F \text{ and } \gamma \text{ in } \Gamma^*\}$$

- **Definition:** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ be a PDA. The *language accepted by empty stack and final state*, denoted $L(M)$, is the set

$$\{w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \varepsilon) \text{ for some } p \text{ in } F\}$$

- **Lemma 1:** Let $L = L_E(M_1)$ for some PDA M_1. Then there exits a PDA M_2 such that $L = L_F(M_2)$.

- **Lemma 2:** Let $L = L_F(M_1)$ for some PDA M_1. Then there exits a PDA M_2 such that $L = L_E(M_2)$.

- **Theorem:** Let L be a language. Then there exits a PDA M_1 such that $L = L_F(M_1)$ if and only if there exists a PDA M_2 such that $L = L_E(M_2)$.

- **Corollary:** The PDAs that accept by empty stack and the PDAs that accept by final state define the same class of languages.

- **Note:** Similar lemmas and theorems could be stated for PDAs that accept by both final state and empty stack.

Back to CFG again:

PDA equivalent to CFG

- **Definition:** Let G = (V, T, P, S) be a CFL. If every production in P is of the form

$$A \rightarrow a\alpha$$

Where A is in V, *a* is in T, and α is in V*, then G is said to be in Greibach Normal Form (GNF).

Only one non-terminal in front.

- **Example:**

$S \rightarrow aAB \mid bB$

$A \rightarrow aA \mid a$

$B \rightarrow bB \mid c$

Language: $(aa^+ + b)b^+c$

- **Theorem:** Let L be a CFL. Then L – {ε} is a CFL.

- **Theorem:** Let L be a CFL not containing {ε}. Then there exists a GNF grammar G such that L = L(G).

- **Lemma 1:** Let *L* be a CFL. Then there exists a PDA *M* such that $L = L_E(M)$.

- **Proof:** Assume without loss of generality that ε is not in *L*. The construction can be modified to include ε later.

  Let G = (V, T, P, S) be a CFG, and assume without loss of generality that *G* is in GNF. Construct M = (Q, Σ, Γ, δ, q, z, Ø) where:

  Q = {q}

  Σ = T

  Γ = V

  z = S

  δ: for all *a* in Σ and *A* in Γ, δ(q, a, A) contains (q, γ)

  if $A \rightarrow a\gamma$ is in *P* or rather:

  $\delta(q, a, A) = \{(q, \gamma) \mid A \rightarrow a\gamma$ is in *P* and γ is in Γ*},

  for all *a* in Σ and *A* in Γ

- For a given string *x* in Σ* , *M* will attempt to simulate a leftmost derivation of *x* with *G*.

- **Example #1:** Consider the following CFG in GNF.

$S \rightarrow aS$
$S \rightarrow a$

G is in GNF
$L(G) = a^+$

Construct M as:

$Q = \{q\}$
$\Sigma = T = \{a\}$
$\Gamma = V = \{S\}$
$z = S$

$\delta(q, a, S) = \{(q, S), (q, \varepsilon)\}$
$\delta(q, \varepsilon, S) = \emptyset$

- *Is δ complete?*

- **Example #2:** Consider the following CFG in GNF.

(1) S –> aA
(2) S –> aB
(3) A –> aA
(4) A –> aB

G is in GNF
$L(G) = a^+ b^+$ // This looks ok to me, one, two or more *a*'s in the start

(5) B –> bB
(6) B –> b

[Can you write a simpler equivalent CFG? Will it be GNF?]

Construct M as:
Q = {q}
Σ = T = {a, b}
Γ = V = {S, A, B}
z = S

(1) δ(q, a, S) = {(q, A), (q, B)} From productions #1 and 2, S->aA, S->aB
(2) δ(q, a, A) = {(q, A), (q, B)} From productions #3 and 4, A->aA, A->aB
(3) δ(q, a, B) = Ø
(4) δ(q, b, S) = Ø
(5) δ(q, b, A) = Ø
(6) δ(q, b, B) = {(q, B), (q, ε)} From productions #5 and 6, B->bB, B->b
(7) δ(q, ε, S) = Ø
(8) δ(q, ε, A) = Ø
(9) δ(q, ε, B) = Ø

Is δ complete?

- For a string w in L(G) the PDA M will simulate a leftmost derivation of w.
  - If w is in L(G) then $(q, w, z_0) \vdash^* (q, \varepsilon, \varepsilon)$
  - If $(q, w, z_0) \vdash^* (q, \varepsilon, \varepsilon)$ then w is in L(G)

- Consider generating a string using G. Since G is in GNF, each sentential form in a *leftmost* derivation has form:

$$\Rightarrow t_1 t_2 \ldots t_i A_1 A_2 \ldots A_m$$

terminals non-terminals

- And each step in the derivation (i.e., each application of a production) adds a terminal and some non-terminals.

$$A_1 \rightarrow t_{i+1} \alpha$$

$$\Rightarrow t_1 t_2 \ldots t_i t_{i+1} \alpha A_1 A_2 \ldots A_m$$

- Each transition of the PDA simulates one derivation step. Thus, the i^{th} step of the PDAs' computation corresponds to the i^{th} step in a corresponding leftmost derivation with the grammar.

- After the i^{th} step of the computation of the PDA, $t_1 t_2 \ldots t_{i+1}$ are the symbols that have already been read by the PDA and $\alpha A_1 A_2 \ldots A_m$ are the stack contents.

- For each leftmost derivation of a string generated by the grammar, there is an equivalent accepting computation of that string by the PDA.

- Each sentential form in the leftmost derivation corresponds to an instantaneous description in the PDA's corresponding computation.

- For example, the PDA instantaneous description corresponding to the sentential form:

$$=> t_1 t_2 \ldots t_i A_1 A_2 \ldots A_m$$

would be:

$$(q, t_{i+1} t_{i+2} \ldots t_n, A_1 A_2 \ldots A_m)$$

- **Example:** Using the grammar from example #2:

S => aA (1)
=> aaA (3)
=> aaaA (3)
=> aaaaB (4)
=> aaaabB (5)
=> aaaabb (6)

Grammar:
(1) S –> aA
(2) S –> aB
(3) A –> aA
G is in GNF
(4) A –> aB
L(G) = a^+b^+
(5) B –> bB
(6) B –> b

- The corresponding computation of the PDA:

(1) δ(q, a, S) = {(q, A), (q, B)}
(2) δ(q, a, A) = {(q, A), (q, B)}
(3) δ(q, a, B) = Ø
(4) δ(q, b, S) = Ø
(5) δ(q, b, A) = Ø
(6) δ(q, b, B) = {(q, B), (q, ε)}
(7) δ(q, ε, S) = Ø
(8) δ(q, ε, A) = Ø
(9) δ(q, ε, B) = Ø

(rule#)/right-side#

- (q, aaaabb, S) |— (q, aaabb, A) (1)/1
|— (q, aabb, A) (2)/1
|— (q, abb, A) (2)/1
|— (q, bb, B) (2)/2
|— (q, b, B) (6)/1
|— (q, ε, ε) (6)/2

  - String is read
  - Stack is emptied
  - Therefore the string is accepted by the PDA

- **Another Example:** Using the PDA from example #2:

(q, aabb, S)	⊢ (q, abb, A)	(1)/1
	⊢ (q, bb, B)	(2)/2
	⊢ (q, b, B)	(6)/1
	⊢ (q, ε, ε)	(6)/2

- The corresponding derivation using the grammar:

S	=> aA	(1)
	=> aaB	(4)
	=> aabB	(5)
	=> aabb	(6)

- **Example #3:** Consider the following CFG in GNF.

(1) S –> aABC
(2) A –> a G is in GNF
(3) B –> b
(4) C –> cAB
(5) C –> cC *Language?*

$$aab\ cc^*\ ab$$

Construct M as:

Q = {q}
Σ = T = {a, b, c}
Γ = V = {S, A, B, C}
z = S

(1) δ(q, a, S) = {(q, ABC)} S->aABC
(2) δ(q, a, A) = {(q, ε)} A->a
(3) δ(q, a, B) = Ø
(4) δ(q, a, C) = Ø
(5) δ(q, b, S) = Ø
(6) δ(q, b, A) = Ø
(7) δ(q, b, B) = {(q, ε)} B->b
(8) δ(q, b, C) = Ø
(9) δ(q, c, S) = Ø
(10) δ(q, c, A) = Ø
(11) δ(q, c, B) = Ø
(12) δ(q, c, C) = {(q, AB), (q, C))} // C->cAB|cC
(13) δ(q, ε, S) = Ø
(14) δ(q, ε, A) = Ø

(15) δ(q, ε, B) = Ø
(8) δ(q, b, C) = Ø (16) δ(q, ε, C) =

- **Notes:**
  - Recall that the grammar *G* was required to be in GNF before the construction could be applied.
  - As a result, it was assumed at the start that ε was not in the context-free language *L*.
  - What if ε is in *L*? You need to add ε back.

- **Suppose ε is in L:**

1) First, let L' = L – {ε}

Fact: If *L* is a CFL, then L' = L – {ε} is a CFL.

By an earlier theorem, there is GNF grammar *G* such that L' = L(G).

2) Construct a PDA *M* such that L' = $L_E(M)$

How do we modify *M* to accept ε?

Add δ(q, ε, S) = {(q, ε)}? *NO!!*

- **Counter Example:**

  Consider L = {ε, b, ab, aab, aaab, …}= ε + a*b Then L' = {b, ab, aab, aaab, …} = a*b

- **The GNF CFG for L':**

  P:

  (1) S –> aS

  (2) S –> b

- **The PDA M Accepting L':**

  Q = {q}

  Σ = T = {a, b}

  Γ = V = {S}

  z = S

  δ(q, a, S) = {(q, S)}

  δ(q, b, S) = {(q, ε)}

  δ(q, ε, S) = Ø

How to add ε to L'now?

$\delta(q, a, S) = \{(q, S)\}$
$\delta(q, b, S) = \{(q, \varepsilon)\}$
$\delta(q, \varepsilon, S) = \emptyset$

•If $\delta(q, \varepsilon, S) = \{(q, \varepsilon)\}$ is added then:

L(M) = {ε, a, aa, aaa, …, b, ab, aab, aaab, …}, wrong!

It is like, S -> aS | b | ε

which is wrong!

Correct grammar should be:

(0) S_1 -> ε | S, with new starting non-terminal S_1

(1) S –> aS

(2) S –> b

For PDA, add a new *Stack-bottom symbol* S_1, with new transitions:

$\delta(q, \varepsilon, S_1) = \{(q, \varepsilon), (q, S)\}$, where *S* was the previous stack-bottom of *M*

Alternatively, add a new *start* state q' with transitions:

$\delta(q', \varepsilon, S) = \{(q', \varepsilon), (q, S)\}$

- **Lemma 1:** Let *L* be a CFL. Then there exists a PDA *M* such that $L = L_E(M)$.

- **Lemma 2:** Let *M* be a PDA. Then there exists a CFG grammar *G* such that $L_E(M) = L(G)$.
  - *Can you prove it?*
  - *First step would be to transform an arbitrary PDA to a single state PDA!*

- **Theorem:** Let *L* be a language. Then there exists a CFG *G* such that $L = L(G)$ iff there exists a PDA *M* such that $L = L_E(M)$.

- **Corollary:** The PDAs define the CFLs.

Sample CFG to GNF transformation:

- 0^n1^n, n>=1
- S -> 0S1 | 01

- GNF:
- S -> 0SS_1 | 0S_1
- S_1 -> 1

- *Note: in PDA the symbol S will float on top, rather than stay at the bottom!*
- *Acceptance of string by removing last S_1 at stack bottom*

Ignore this slide

- How about language like: (((())()))(), nested

$M = (Q, \Sigma, \Gamma, \delta, q_1, \#, \emptyset)$

δ:

(1) $\delta(q_1, (, \#) = \{(q_1, L\#)\}$

(2) $\delta(q_1,), \#) = \emptyset$ // illegal, string rejected

(3) $\delta(q_1, (, L) = \{(q_1, LL)\}$

(4) $\delta(q_1,), L) = \{(q_2, \varepsilon)\}$

(5) $\delta(q_2,), L) = \{(q_2, \varepsilon)\}$

(6) $\delta(q_2, (, L) = \{(q_1, LL)\}$ // not balanced yet, but start back anyway

(7) $\delta(q_2, (, \#) = \{(q_1, L\#)\}$ // start afresh again

(8) $\delta(q_2, \varepsilon, \#) = \{(q_2, \varepsilon)\}$ // end of string & stack hits bottom, accept

(9) $\delta(q_1, \varepsilon, \#) = \{(q_1, \varepsilon)\}$ // special rule for empty string

(10) $\delta(q_1, \varepsilon, L) = \emptyset$ // illegal, end of string but more L in stack

Total number of transitions? Verify all carefully.