# MAT5B07
# BASIC MATHEMATICAL ANALYSIS

*BY*

**ALPHY JOSE**

**LITTLE FLOWER COLLEGE, GURUVAYOOR**

# COMPLEX ANALYSIS

## SECTION 4: VECTORS AND MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point $(x, y)$ that represents $z$ in the complex plane. In fact, we often refer to $z$ as the point $z$ or the vector $z$. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

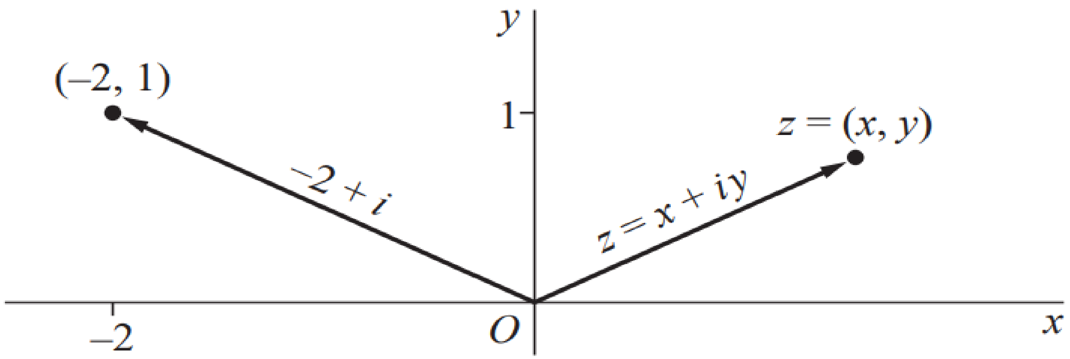


**FIGURE 2**

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 3.

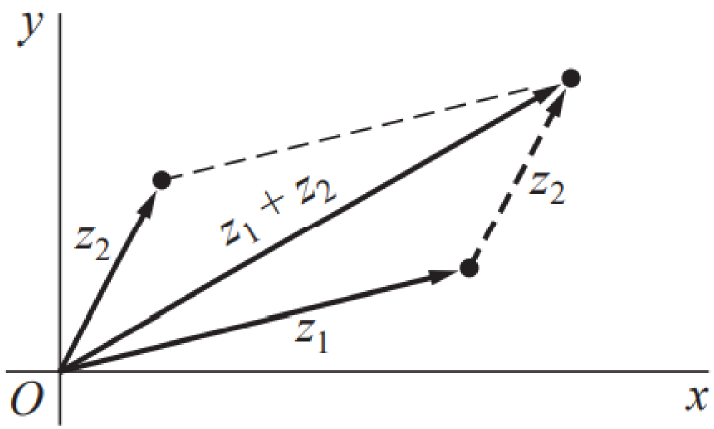


**FIGURE 3**

Although the product of two complex numbers $z_1$ and $z_2$ is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for $z_1$ and $z_2$. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The *modulus,* or absolute value, of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}. \tag{1}$$

Geometrically, the number $|z|$ is the distance between the point $(x, y)$ and the origin, or the length of the radius vector representing $z$. It reduces to the usual absolute value in the real number system when $y = 0$. Note that while *the inequality $z_1 < z_2$ is meaningless unless both $z_1$ and $z_2$ are real,* the statement $|z_1| < |z_2|$ means that the point $z_1$ is closer to the origin than the point $z_2$ is.

# EQUATION OF A CIRCLE

The complex numbers $z$ corresponding to the points lying on the circle with center $z_0$ and radius $R$ thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

**EXAMPLE 2.** The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

(2) $$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2.$$

(3) $$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \quad \text{and} \quad \operatorname{Im} z \le |\operatorname{Im} z| \le |z|.$$

the *triangle inequality,*

(4) $$|z_1 + z_2| \le |z_1| + |z_2|.$$

# SECTION 5: COMPLEX CONJUGATES

The *complex conjugate,* or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is,

(1) $$\bar{z} = x - iy.$$

The number $\bar{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point $(x, y)$ representing $z$ (Fig. 5). Note that

$$\bar{\bar{z}} = z \quad \text{and} \quad |\bar{z}| = |z|$$

for all $z$.

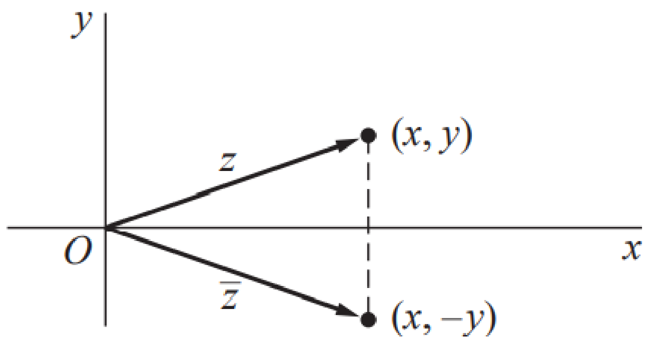


**FIGURE 5**

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

So the conjugate of the sum is the sum of the conjugates:

(2) $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}.$$

In like manner, it is easy to show that

(3) $$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2},$$

(4) $$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2},$$

and

(5) $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0).$$

The sum $z + \bar{z}$ of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ is the real number $2x$, and the difference $z - \bar{z}$ is the pure imaginary number $2iy$. Hence

(6) $$\operatorname{Re} z = \frac{z + \bar{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

# SECTION 6 EXPONENTIAL FORM

Let $r$ and $\theta$ be polar coordinates of the point $(x, y)$ that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number $z$ can be written in *polar form* as

(1) $$z = r(\cos\theta + i\sin\theta).$$

If $z = 0$, the coordinate $\theta$ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

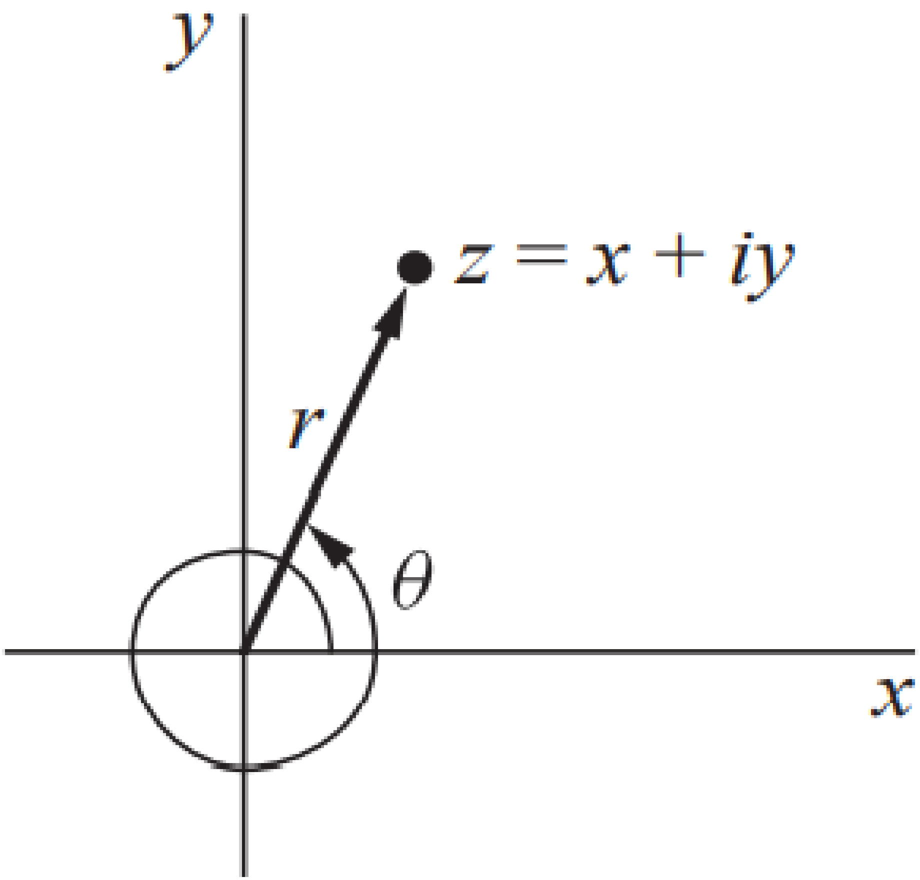


FIGURE 6

- In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z ; that is, $\mathbf{r = |z|}$.
- The real number $\boldsymbol{\theta}$ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6).

- As in calculus, $\theta$ has an infinite number of possible values, including negative ones, that differ by integral multiples of $2\pi$. Those values can be determined from the equation **tan $\theta$ = y/x**, where the quadrant containing the point corresponding to z must be specified.
- Each value of $\theta$ is called an **argument of z**, and the set of all such values is denoted by **arg z.**

- The principal value of arg z, denoted by $\boldsymbol{Arg\ z}$, is that unique value $\Theta$ such that $-\pi < \Theta \leq \pi$.
- $\arg z = Arg\ z + 2n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$.
- When $z$ is a negative real number, $Arg\ z$ has value π, not −π.

- The numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity

$$e^{i\pi} = -1, \quad e^{-i\pi/2} = -i, \quad \text{and} \quad e^{-i4\pi} = 1.$$

# SECTION 7: PRODUCTS & POWERS IN EXPONENTIAL FORM

$$
\begin{aligned}
e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\
&= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}.
\end{aligned}
$$

Thus, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the product $z_1 z_2$ has exponential form

(1) $$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}.$$

Furthermore,

(2) $$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \cdot \frac{e^{i\theta_1} e^{-i\theta_2}}{e^{i\theta_2} e^{-i\theta_2}} = \frac{r_1}{r_2} \cdot \frac{e^{i(\theta_1-\theta_2)}}{e^{i0}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Note how it follows from expression (2) that the inverse of any nonzero complex number $z = re^{i\theta}$ is

(3) $$z^{-1} = \frac{1}{z} = \frac{1e^{i0}}{re^{i\theta}} = \frac{1}{r} e^{i(0-\theta)} = \frac{1}{r} e^{-i\theta}.$$

**EXAMPLE 1.** In order to put $(\sqrt{3}+1)^7$ in rectangular form, one need only write

$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3}+i).$$

Finally, we observe that if $r = 1$, equation (4) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{6}$$

this is known as *de Moivre's formula*. The following example uses a special case of it.

**EXAMPLE 2.** Formula (6) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta,$$

or

$$\cos^2\theta - \sin^2\theta + i2\sin\theta \quad \cos\theta = \cos 2\theta + i\sin 2\theta.$$

By equating real parts and then imaginary parts here, we have the familiar trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

# SECTION 8: ARGUMENTS OF PRODUCTS AD QUOTIENTS

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the expression

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \tag{1}$$

in Sec. 7 can be used to obtain an important identity involving arguments:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{2}$$

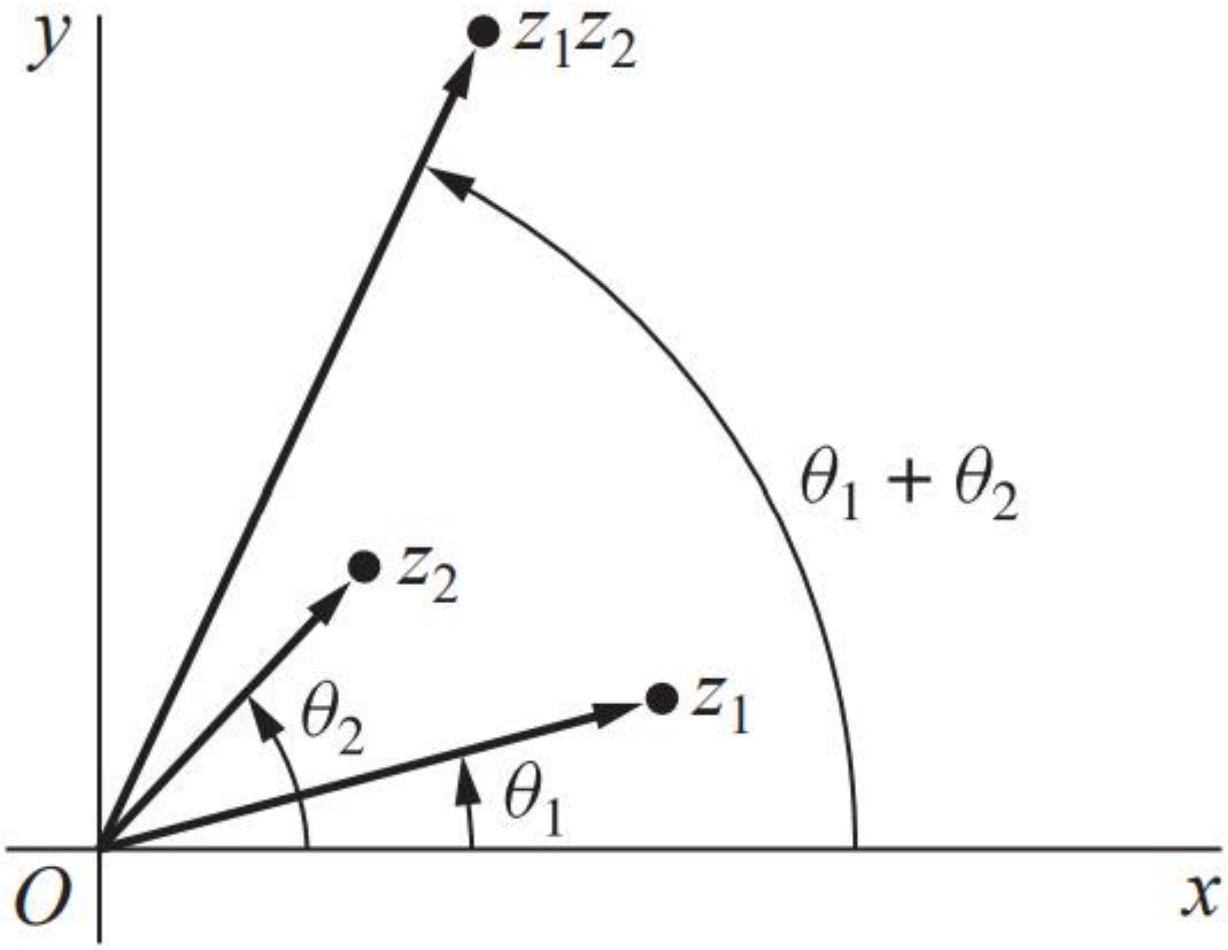


FIGURE 9

# PROOF

Let $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

ie., $arg\, z_1 = \theta_1$ and $arg\, z_2 = \theta_2$.

Expression (1) then tells us that $\theta_1 + \theta_2$ is a value of $arg\, z_1 z_2$ .

But it may not be the case if we replace arg z by Arg z.

# EXAMPLE

Let $z_1 =$ -1 and $z_2 =$ i

Then $z_1 z_2 =$ -i

$$Arg\ z_1 = Arg\ (-1) = \pi$$

$$Arg\ z_2 = Arg\ (i) = \frac{\pi}{2}$$

$$Arg\ (z_1 z_2) = Arg\ (-i) = -\frac{\pi}{2}$$

But $Arg\ z_1 + Arg\ z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$

$$Arg\ (z_1 z_2) \neq Arg\ z_1 + Arg\ z_2$$

Statement (2) tells us that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1 z_2^{-1}) = \arg z_1 + \arg(z_2^{-1});$$

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

$$\arg(z_2^{-1}) = -\arg z_2.$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

# THANK YOU

# HAVE A NICE DAY