MAT5B07

BASIC MATHEMATICAL ANALYSIS

COMPLEX ANALYSIS

BY

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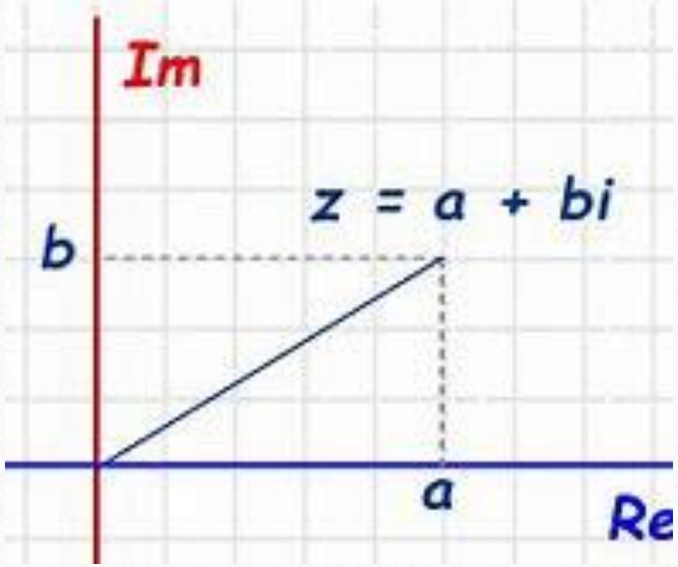
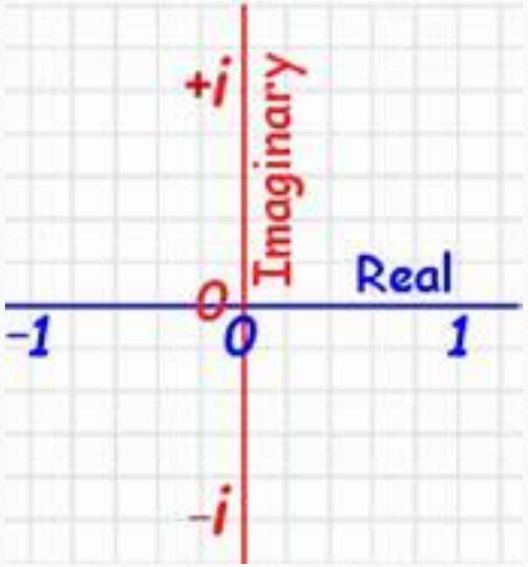
CHAPTER 1
COMPLEX NUMBERS

SECTION 1 : SUM AND PRODUCTS

- Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the complex plane, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line.





- When real numbers x are displayed as points $(x, 0)$ on the real axis, it is clear that the set of complex numbers includes the real numbers as a subset.

- Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called pure imaginary numbers when $y \neq 0$.
- The y axis is then referred to as the imaginary axis
- We denote a complex number (x, y) by z and write $z = (x, y)$

- We denote a complex number (x, y) by z
  ie, $\mathrm{z} = (x, y)$

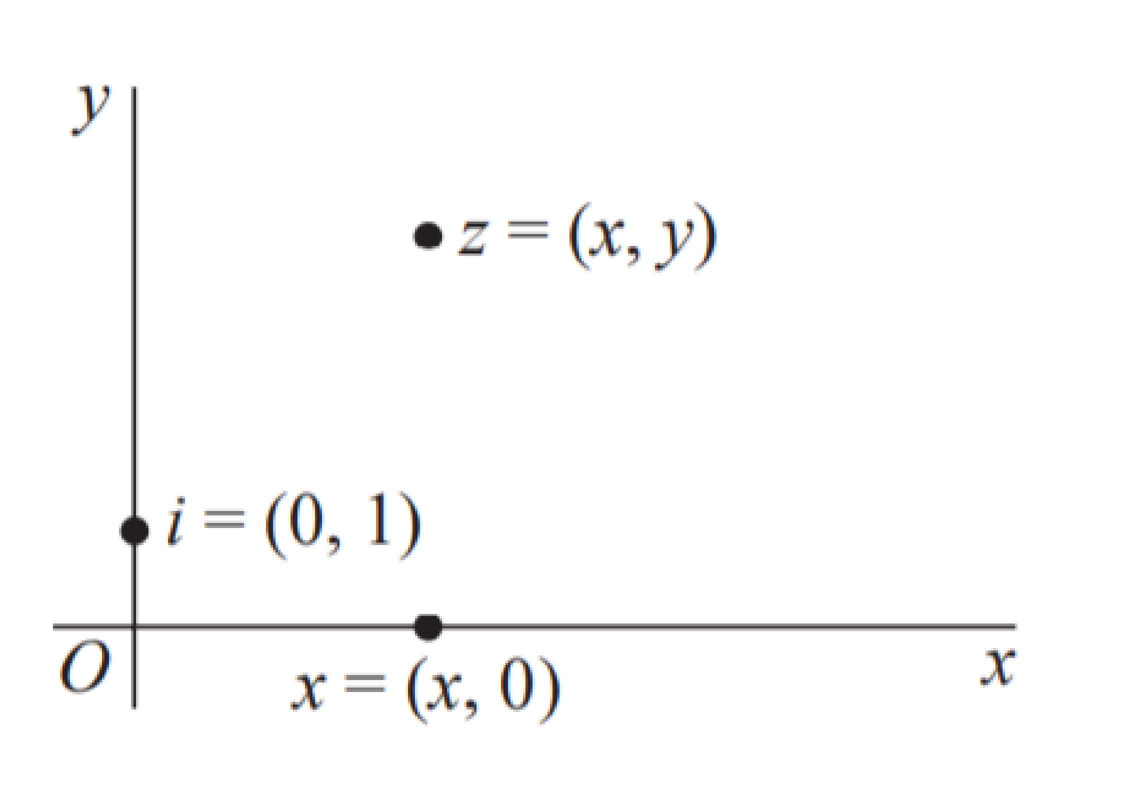
- The real numbers x and y are known as the **real part** and **imaginary part** of z respectively; and we write

$$x = Re\, z, y = Im\, z$$

NOTES

- Two complex numbers z_1 and z_2 are equal whenever they have the same real parts and the same imaginary parts.
- The statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex plane or z plane

Complex plane or z plane



The *sum* $z_1 + z_2$ and *product* $z_1 z_2$ of two complex numbers

$$z_1 = (x_1, y_1) \quad \text{and} \quad z_2 = (x_2, y_2)$$

are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2). \tag{4}$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and if we think of a real number as either x or $(x, 0)$ and let i denote the pure imaginary number $(0,1)$, as shown in Fig. 1, it is clear that*

$$z = x + iy. \tag{5}$$

Also, with the convention that $z^2 = zz$, $z^3 = z^2 z$, etc., we have

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

$$i^2 = -1. \tag{6}$$

Because $(x, y) = x + iy$, definitions (3) and (4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \tag{7}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \tag{8}$$

REMARK

- Any complex number times zero is zero.
- More precisely, $z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0$ for any $z = x + iy$.

SECTION 2
BASIC ALGEBRAIC PROPERTIES

- Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them.

LAWS OF ADDITION AND MULTIPLICATION

(1) The commutative laws

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1$$

(2) The associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad z_1\ z_2\ z_3 = z_1\ z_2\ z_3$$

(3) The distributive law

$$z(z_1 + z_2) = zz_1 + zz_2$$

PROOF

Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

Then $z_1 + z_2 = (x_1 + x_2\,, y_1 + y_2)$

$=(x_2 + x_1\,, y_2 + y_1)$

$= z_2 + z_1$

Hence commutativity of addition of complex numbers is proved.

NOTES

- According to the commutative law for multiplication, $iy = yi$. Hence one can write $z = x + yi$ instead of $z = x + iy$.
- Because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1 z_2 z_3$ is well defined without parentheses, as is the case with real numbers.

IDENTITIES

- The additive identity in the complex number system is $0 = (0, 0)$ such that $z + 0 = z$ for every complex number z.
- The multiplicative identity in the complex number system is $1 = (1, 0)$ such that $z \cdot 1 = z$ for every complex number z.
- 0 and 1 are the only complex numbers with these properties.

ADDITIVE INVERSE

- Associated with each complex number $z = (x, y)$ there exists an additive inverse $-z = (-x, -y)$, satisfying the equation $z + (-z) = 0$.
- Moreover, there is only one additive inverse for any given z, since the equation $(x, y) + (u, v) = (0, 0)$ implies that $u = -x$ and $v = -y$.

MULTIPLICATIVE INVERSE

- For any nonzero complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$
- To find z^{-1}: We have to find real numbers u and v, expressed in terms of x and y, such that $(x, y).(u, v) = (1, 0)$

ie., $(xu - yv, \ \ yu + xv) = (1, 0)$

ie., $xu - yv = 1, yu + xv = 0$

By simple computation, we can find

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

$$(6) \qquad z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \qquad (z \neq 0).$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression for z^{-1}

RESULT 1: If a product $z_1 z_2$ is zero, then so is at least one of the factors z_1 and z_2

Proof: Suppose that $z_1 z_2 = 0$ and $z_1 \neq 0$.

Then ${z_1}^{-1}$exists.

Hence $z_2 = z_2 . 1 = z_2 . (z_1 {z_1}^{-1})$

$= (z_2 . z_1){z_1}^{-1} = (z_1 . z_2){z_1}^{-1} = 0 . {z_1}^{-1} = 0$

Similarly, $z_1 z_2 = 0$ and $z_2 \neq 0 \implies\implies z_1 = 0$

That is, if $z_1 z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both of the numbers z_1 and z_2 are zeros.

Another way to state this result is that **if two complex numbers z_1 and z_2 are nonzero, then so is their product** $z_1 z_2$

Subtraction and division are defined in terms of additive and multiplicative inverses:

(1) $$z_1 - z_2 = z_1 + (-z_2),$$

(2) $$\frac{z_1}{z_2} = z_1 z_2^{-1} \qquad (z_2 \neq 0).$$

Thus, in view of expressions (5) and (6) in Sec. 2,

(3) $$z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2)$$

and

(4) $$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right) \qquad (z_2 \neq 0)$$

when $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Using $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one can write expressions (3) and (4) here as

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{5}$$

and

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0). \tag{6}$$

Although expression (6) is not easy to remember, it can be obtained by writing (see Exercise 7)

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}, \tag{7}$$

EXERCISES

1. Verify that

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$; *(b)* $(2, -3)(-2, 1) = (-1, 8)$;

(c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

2. Show that

(a) $\operatorname{Re}(iz) = -\operatorname{Im} z$; *(b)* $\operatorname{Im}(iz) = \operatorname{Re} z$.

3. Show that $(1 + z)^2 = 1 + 2z + z^2$.

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

5. Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

6. Verify

(a) the associative law for addition of complex numbers, stated at the beginning of Sec. 2;

(b) the distributive law (3), Sec. 2.

7. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

8. *(a)* Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

9. Use $-1 = (-1, 0)$ and $z = (x, y)$ to show that $(-1)z = -z$.

10. Use $i = (0, 1)$ and $y = (y, 0)$ to verify that $-(iy) = (-i)y$. Thus show that the additive inverse of a complex number $z = x + iy$ can be written $-z = -x - iy$ without ambiguity.

11. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

SECTION 3: FURTHER PROPERTIES

- In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described multiplying out the products in the numerator and denominator on the right, and then using the property

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \qquad (z_3 \neq 0).$$

EXAMPLE. The method is illustrated below:

$$\frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{(2-3i)(2+3i)} = \frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

BASIC EQUATIONS

(1) $$z_1 - z_2 = z_1 + (-z_2),$$

(2) $$\frac{z_1}{z_2} = z_1 z_2^{-1} \qquad (z_2 \neq 0).$$

Thus, in view of expressions (5) and (6) in Sec. 2,

(3) $$z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2)$$

and

(4) $$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right) \qquad (z_2 \neq 0)$$

$$(9) \qquad \frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0),$$

which is equation (2) when $z_1 = 1$. Relation (9) enables us, for instance, to write equation (2) in the form

$$(10) \qquad \frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0).$$

Also, by observing that (see Exercise 3)

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \qquad (z_1 \neq 0,\ z_2 \neq 0),$$

and hence that $z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1}$, one can use relation (9) to show that

$$(11) \qquad \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1} = \frac{1}{z_1 z_2} \qquad (z_1 \neq 0,\ z_2 \neq 0).$$

Another useful property, to be derived in the exercises, is

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \qquad (z_3 \neq 0,\ z_4 \neq 0). \tag{12}$$

Finally, we note that the *binomial formula* involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \qquad (n = 1, 2, \ldots) \tag{13}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof is left as an exercise.

EXERCISES

1. Reduce each of these quantities to a real number:

$$(a)\ \frac{1+2i}{3-4i}+\frac{2-i}{5i}; \qquad (b)\ \frac{5i}{(1-i)(2-i)(3-i)}; \qquad (c)\ (1-i)^4.$$

Ans. *(a)* $-2/5$; *(b)* $-1/2$; *(c)* -4.

2. Show that

$$\frac{1}{1/z}=z \qquad (z\neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

4. Prove that if $z_1z_2z_3=0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1z_2)z_3=0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \qquad (z_3 \neq 0, z_4 \neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \qquad (z_2 \neq 0, z \neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

Suggestion: When $n = m + 1$, write

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1) \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^{k+1} z_2^{m-k}$$

and replace k by $k - 1$ in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

THANK YOU
HAVE A NICE DAY