# MITTAG-LEFFLER DISTRIBUTIONS

## MINOR RESEARCH PROJECT
## (XII$^{th}$ PLAN)

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**SUMMARY OF THE PROJECT**

The fundamental area of Probability Theory attracted many researchers.One important aspect of Probability Theory is to provide probability distributions to random variables defined over uncertain situations.Scientists in the field of probability distributions introduced numerous distributions like binomial, Poisson,exponential, geometric etc. Among them, normal distribution, the most important provides the best fit in many types of real life data and,in a number of problems, the limiting distribution is normal. One important parametric family among life distributions is the exponential distribution, which plays a central role within the class of all life distributions.There are numerous situations where the exponential law is unsuitable to model. Recently much effort has been focused on the study of distributions that are heavy tailed as compared to exponential distribution. Pillai (1990) introduced Mittag –Leffler distribution as a generalization to exponential distribution.The data consisting of commodity prices, daily stock returns, foreign currency exchange rates and other financial data can adequately modeled using Mittag-Leffler distribution.

The Mittag-Leffler function was introduced by Swedish Mathematician Gosta Mittag-Leffler in 1903 (see, Mittag-Leffler(1903)) in connection with his method of summations of some divergent series.The function $E_\alpha$ (u) = $\sum_{k=0}^{\infty} \frac{u^k}{\Gamma(1+\alpha k)}$ , u ∈ $(0,\infty)$ is known as Mittag Leffler function. It arises as the solution of certain boundary value problems involving fractional differential equation. During the various developments of fractional calculus in the last two decades, this function has gained importance and popularity on account of its vast applications in the field of Science and technology (see, Mathai (2010) and Pillai (1990)).

Feller (1971) showed that the Laplace transform of $E_\alpha(-x^\alpha)$ for $0 \leq \alpha \leq 1$ is $\frac{\lambda^{\alpha-1}}{1+\lambda^\alpha}$ , $\lambda \geq 0$. But $E_\alpha(-x^\alpha)$ is not a probability distribution. Pillai (1990) showed that $F_\alpha$ (x) = 1- $E_\alpha(-x^\alpha)$ is a distribution function. Hence he named $F_\alpha$ (x) as Mittag- Leffler (ML) distribution. We have $F_\alpha$ (x) = $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{\alpha k}}{\Gamma(1+\alpha k)}$ , $0< \alpha \leq 1$, x$\geq 0$ and the corresponding density function is $f_\alpha$ (x) = $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{\alpha k-1}}{\Gamma(\alpha k)}$.

The report of the project is presented in five chapters. The first Chapter is Introduction and presents the need worthy materials for subsequent discussions. Univariate continuous ML distribution and its various properties are reviewed in Chapter 2. Also, we review the literature on generalized Mittag-Leffler, semi-Mittag-Leffler and geometric Mittag-Leffler distributions. The review is based on the works of Pillai (1990), Jayakumar and Pillai (1993,1996), Pillai and Jayakumar (1994), Jayakumar and Suresh (2003), Jayakumar and Ajitha (2003), Jose et al. (2010), Mathai (2010), Lin (1998, 2001) and Hulliet (2016).

A discrete analogue of ML distribution, namely Discrete Mittag-Leffler (DML) distribution was introduced and studied by Pillai and Jayakumar (1995). In Chapter 3,we review the mathematical origin and properties of DML distribution. Another discrete analogue of ML distribution was recently introduced by Chakraborty and Ong (2017). Various aspects of this distribution, known as, Mittag-Leffler Function Distribution(MLFD) is also reviewed in this Chapter. Note that the MLFD is a generalization of Poisson distribution. For details, see Mariamma (2018). In this Chapter, we also review the literature on discrete geometric Mittag-Leffler distribution introduced and studied in Jayakumar and Ajitha (2003).

In Chapter 4,we review various aspects of bivariate ML and bivariate DML distributions. The discussion in this Chapter is based on Jayakumar and Mundassery (2006), Mundassery and Jayakumar(2006) and Jayakumar et al. (2010).

Finally, in Chapter 5,we present applications of ML and DML distributions in various areas, especially in autoregressive time series modeling. The discussion in this Chapter is based on Jose et al. (2010), Weron and Kotulski(1996), Jayakumar(2003), Mathai (2010), Chakraborty and Ong (2017) and Mariamma (2017, 2018).

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