Relativity

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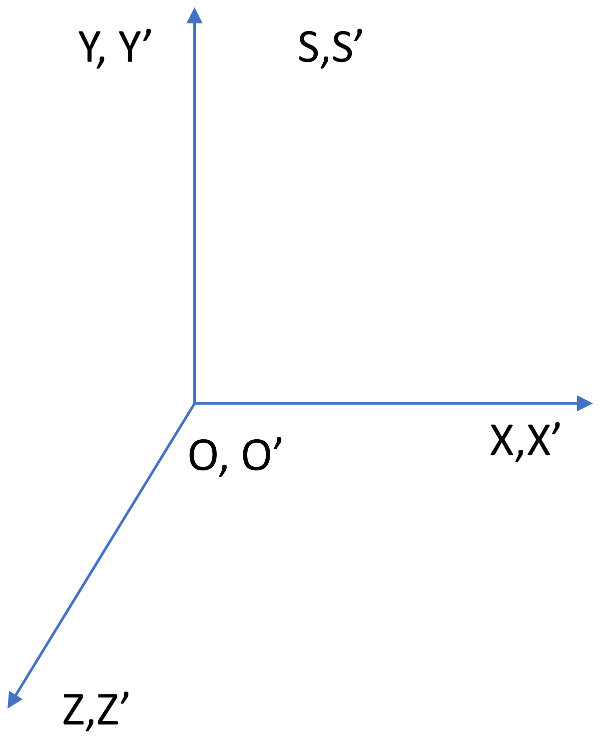
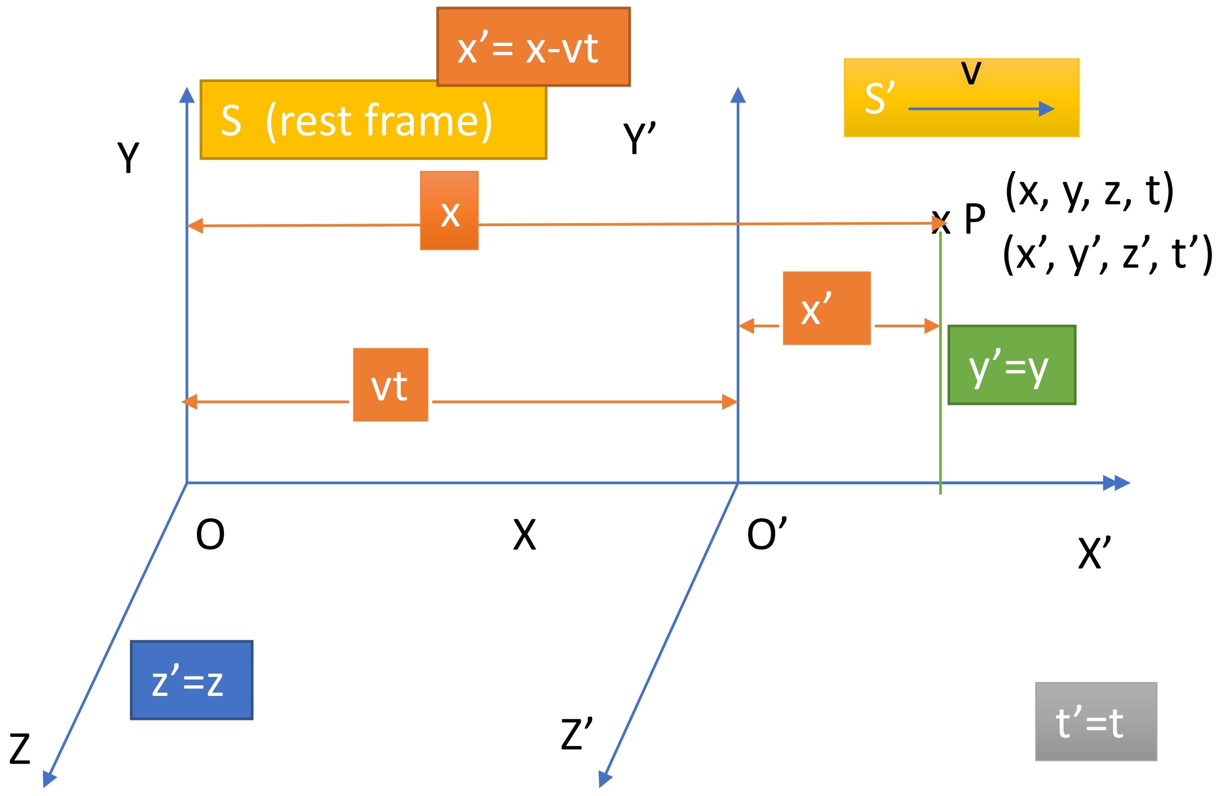
Lorentz transformations

Lorentz derived transformation equations to replace Galilean Transformation equation to give results in agreement with Michelson Morely experiment and also in accordance with the postulates of special theory of relativity. These equations connecting the position and time co-ordinates of an event measured from two inertial frames are known as Lorentz transformations.

Galilean Transformations

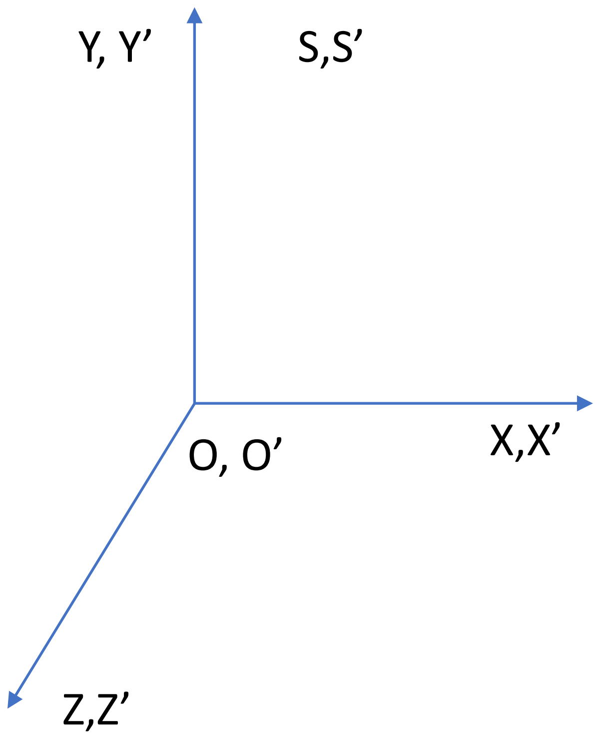
t=t'=0

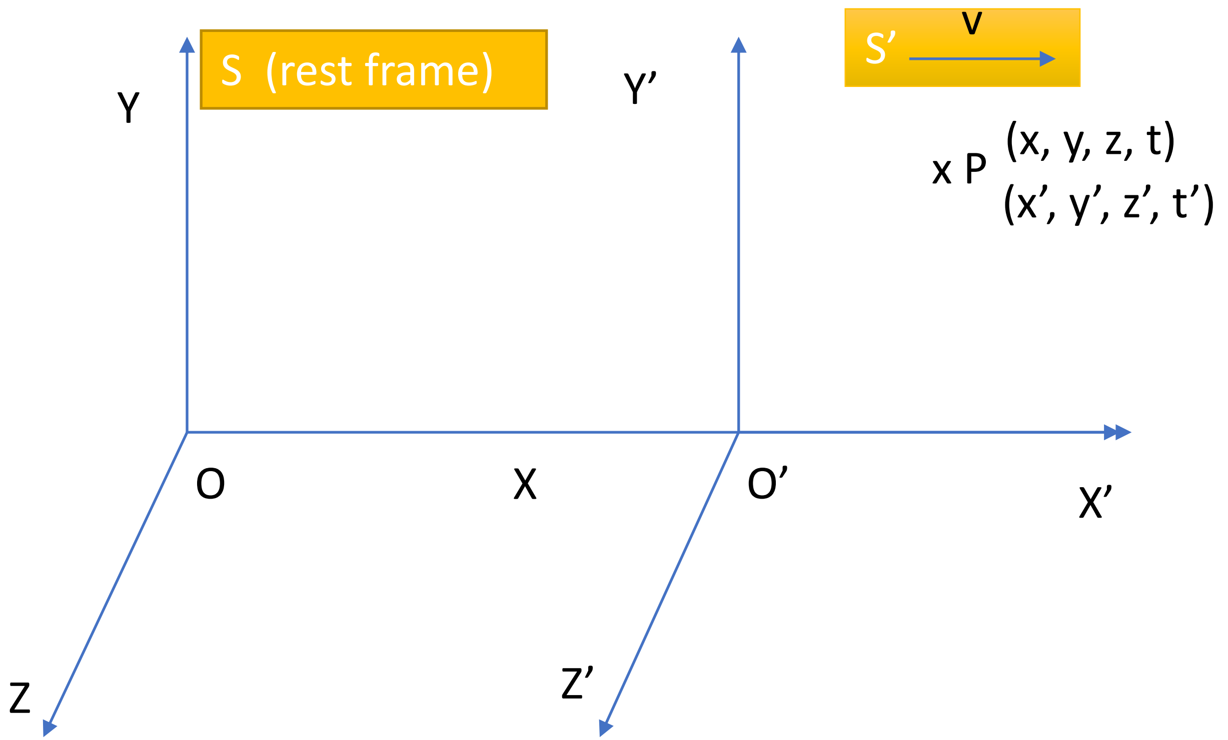




Lorentz Transformations

t=t’=0





Let a pulse of light be produced at the origin at t=0. Suppose the pulse of light reaches at a point P. let the co-ordinates of the event at P in frame S’ be (x’, y’, z’, t’) and corresponding co-ordinates in frame S be (x,y,z,t).

The transformations must be linear to satisfy the first postulate of relativity. A single event in S must appear as a single event in S’ also.

So the transformation equations can be written as

$x' = k(x - vt)$, where k is a constant independent of x and t

y’=y

z’=z

According to first postulate, the motion of S’ relative to S with velocity v is same as the motion of S w.r.t. S’ with velocity –v

Then x=k(x’+vt’)

Substitute for x'. Then we get

x= $k[k(x - vt) + vt']$

x= $k[kx - kvt + vt']$

$$x = k^2x - k^2vt + kvt'$$

$$kvt' = x - k^2x + k^2vt$$

$$kvt' = x(1 - k^2) + k^2vt$$

$$t' = \frac{x(1-k^2)}{kv} + \frac{k^2vt}{kv}$$

$$t' = \frac{x(1-k^2)}{kv} + \text{kt}$$

To find k

Here, we use the second postulate that the velocity of light is constant in both frames. Then

$$x = ct$$
$$x' = ct'$$

$x' = ct'$ can be written as

$$k(x - vt) = c\left[\frac{x(1-k^2)}{kv} + \mathrm{kt}\right]$$

$$kx - kvt = \frac{cx(1-k^2)}{kv} + \mathrm{ckt}$$

Substitute for x

$$kct - kvt = \frac{c^2t(1-k^2)}{kv} + \mathrm{ckt}$$

$$-kvt = \frac{c^2t(1-k^2)}{kv}$$

$$-k^2v^2t = c^2t(1 - k^2)$$
$$k^2c^2t - k^2v^2t = c^2t$$

Divide the equation by t

$$k^2c^2 - k^2v^2 = c^2$$
$$k^2(c^2 - v^2) = c^2$$

$$k^2 = \frac{c^2}{(c^2-v^2)} = \frac{\frac{c^2}{c^2}}{\frac{c^2}{c^2} - \frac{v^2}{c^2}} = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore we get *Lorentz Transformations*

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , y' = y \quad , z' = z$$

In order to get the equation for t', substitute for k in $t' = \frac{x(1-k^2)}{kv} + kt$

We get $t' = \frac{x\left(1 - \frac{1}{1-\frac{v^2}{c^2}}\right)}{\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}} + \frac{t}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x\sqrt{1-\frac{v^2}{c^2}}}{v}\left(1 - \frac{1}{1-\frac{v^2}{c^2}}\right) + \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$

$$= \frac{x\sqrt{1-\frac{v^2}{c^2}}}{v}\left(\frac{1-\frac{v^2}{c^2}-1}{1-\frac{v^2}{c^2}}\right) + \frac{t}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x}{v}\left(\frac{-\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right) + \frac{t}{\sqrt{1-\frac{v^2}{c^2}}} = \left(\frac{-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right) + \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

- *Lorentz Transformations*

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , y' = y \quad , z' = z \quad , t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- Inverse Lorentz transformation equations

The Inverse Lorentz transformation equations are obtained by interchanging primed and unprimed quantities and v is replaced by −v.

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , y = y' \quad , z = z' \quad , t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

When $v \ll c$,

$\frac{v}{c} \ll 1.$ *Hence* $\frac{v^2}{c^2}$ *can be neglected*

Hence Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , y' = y \quad , z' = z \quad , t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

reduces to Galilean transformations

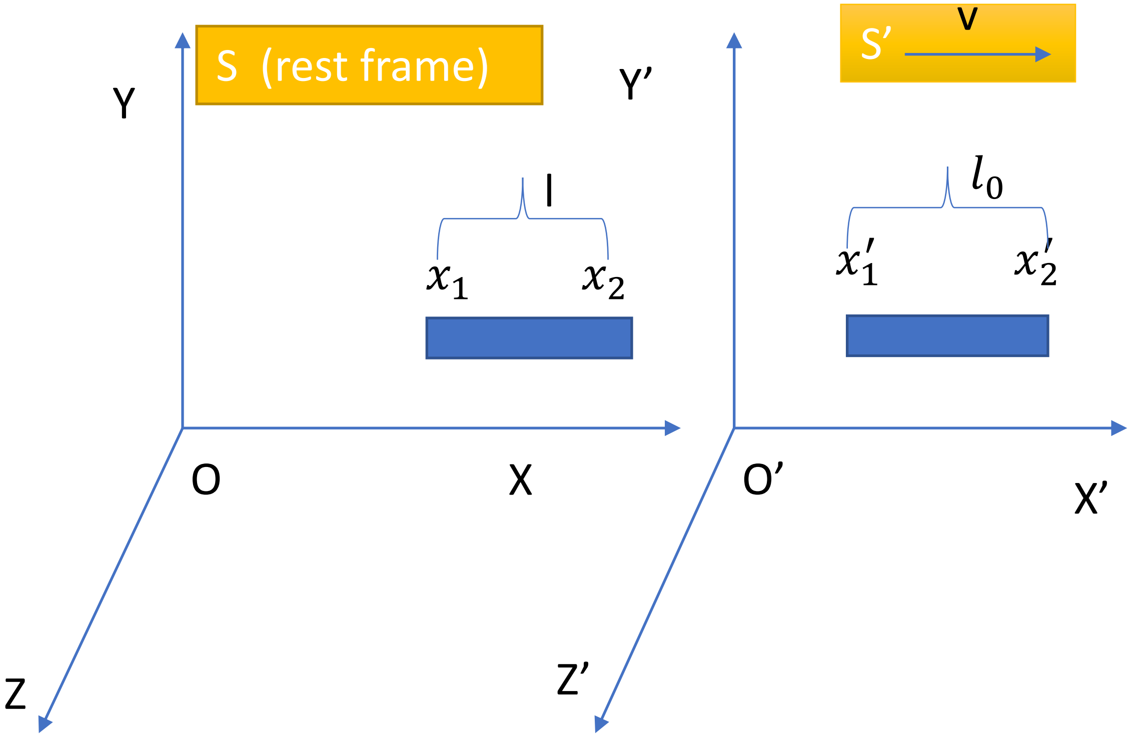
$$x' = x - vt \ , y' = y \ , \quad z' = z \quad , \quad t' = t$$

LENGTH CONTRACTION

Consider two inertial frames of reference S and S'. S is at rest and S' is moving in the positive X-axis direction with uniform velocity v.

A rod is kept in the frame S' along the X-axis. The rod is at rest to an observer in the frame S'. If x_1' and x_2' are the coordinates of the ends of the rod measured in this frame at the same time t, then length of the rod at rest is the proper length of the rod.

$$l_0 = x_2' - x_1'$$



The rod is viewed from the stationary frame S and the coordinates of the ends of the rod with respect to this frame are x_1 and x_2, *then*

$$l = x_2 - x_1$$

Using Lorentz transformations

$x_2' = \frac{x_2 - vt}{\sqrt{1-\frac{v^2}{c^2}}}$ and $x_1' = \frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}$

$\therefore\ l_0 = x_2' - x_1' = \frac{x_2 - vt}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2 - vt - x_1 + vt}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x_2 - x_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{l}{\sqrt{1-\frac{v^2}{c^2}}}$

$\therefore\ l = l_0\sqrt{1-\frac{v^2}{c^2}}$

Since $\sqrt{1-\frac{v^2}{c^2}}$ is always less than unity, $l < l_0$.

The length of moving rod appears to be shortened by a factor $\sqrt{1-\frac{v^2}{c^2}}$ in the direction of its motion. This is known as Lorentz-Fitzgerald contraction.

Since y'=y and z'=z, the dimensions in these directions are unaffected.

The length contraction is observed in particles moving with very high speed.

The measurement of length of bodies are affected by the relative motion between the frames of reference.

The length of an object measured in a frame of reference in which it is at rest relative to an observer is maximum and is called its proper length.

The length of the object appears to be contracted when it is moving with a velocity relative to an observer. The contraction takes place in the direction of motion and its dimensions are unaffected in the perpendicular direction of motion. This is called Lorentz- Fitzgerald contraction.

The amount of contraction can be calculated by Lorentz transformation.

A rocket is 40 metre long on the ground. When it is in flight its length is 38 metre to an observer on the ground. Find the speed of the rocket.

- $l_0 = 40m$ $l=38m$ $v=?$

$$l = l_0\sqrt{1-\frac{v^2}{c^2}} \Rightarrow \frac{l}{l_0} = \sqrt{1-\frac{v^2}{c^2}} \Rightarrow \left(\frac{l}{l_0}\right)^2 = 1-\frac{v^2}{c^2}$$

$$\Rightarrow \frac{v^2}{c^2} = 1-\left(\frac{l}{l_0}\right)^2 \Rightarrow \frac{v}{c} = \sqrt{1-\left(\frac{l}{l_0}\right)^2}$$

$$\Rightarrow v = c\sqrt{1-\left(\frac{l}{l_0}\right)^2} = c\sqrt{1-\left(\frac{38}{40}\right)^2} = 0.312c$$

Effect on time interval-Time dilation

The time interval between two events are also influenced by the relative motion of the frames.

Consider two inertial frames S and S'. Let S be at rest and S’ moving with a velocity v in direction of X- axis. Initially let the two frames coincide at time t = 0.

Let two events occur at a point x' in the frame S' at times t_1' and t_2' as noted on the clock carried by an observer in that frame. The time interval between the two events measured in the frame S' is t_2'- t_1'.

The two events are observed from the stationary frame S and these occur at times $t_1$and t_2 as noted on the clock in this frame. Hence the time interval between the events measured in the frame S = t_2- t_1

Lorentz transformation is $t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$

S' is moving with a velocity v w.r.t. S. It means that S is moving with a velocity -v w.r.t. S'. So have to use the inverse Lorentz transformations.

In the above equation v is changed to –v. t and t' are interchanged. Then

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Applying the above Lorentz inverse equation to the two events in the respective frames S and S', we get

$$t_1 = \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad t_2 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Time interval recorded in the stationary frame S

$$\Delta t = t_2 - t_1 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t_2' + \frac{vx'}{c^2} - t_1' - \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Where $\Delta t_0 = t_2' - t_1'$ is the time interval noted in the moving frame S' and is called proper time interval.

Since $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is always greater than unity, $\Delta t > \Delta t_0$.

The time interval of events observed by a clock in the rest frame appears longer as compared to the time interval recorded by the clock in the frame S'. Hence the clock runs faster in the frame S. At the same time the moving clock runs slow and the time interval appears to be expanded or dilated Due to this effect the clocks in the moving space ships will appear to go slower than the clocks on the surface of earth. This is called time dilation

Experimental Verification of Time Dilation

Time dilation has been verified in the case of elementary particles μ-mesons. π^+Mesons are produced in the atmosphere of earth due to the interaction of cosmic ray with the atmospheric gases. The π^+ Mesons decays in to μ^+ mesons with in a very short time.

The velocity of muons is about 0.998c. The mean life time of μ^+ mesons is $2.2 \times 10^{-6} sec$ in a frame in which they are at rest (Frames of their own). Hence the distance it can travel in this life time is 0.998cX 2.2x 10^{-6} m = 0.653km.

Since muons are produced at a height of 10km above the surface of earth, no Muons are expected to be present on the surface of earth. But experimentally, an appreciable number of muons are detected on the surface of earth.This can be explained on the basis of time dilation.

For an observer on the surface of earth, the mean life of muons will be Δt .

We have $\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.2 \times 10^{-6}}{\sqrt{1-\frac{(0.998c)^2}{c^2}}} = \frac{2.2 \times 10^{-6}}{\sqrt{1-0.998^2}} = 34.80 \times 10^{-6} sec$

That is, the mean life time of μ-mesons is increased due to its high speed. The distance to which μ-mesons can travel in this time is 0.998cX34.80×10^{-6} = 10.4 km

Without time dilation the distance that can be covered by the muons = 0.66km. When there is time dilation the distance increases to 10.4km.

This explains the presence of μ-mesons near the surface of earth and thus verifies time dilation.

A particle with mean proper life time of $\mathbf{2 \times 10^{-6}}$sec moves through the laboratory with a speed of 0.99c. Calculate its life time as measured by an observer in laboratory.

$\Delta t_0 = 2 \times 10^{-6}$sec　　　　v=0.99c

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2 \times 10^{-6}}{\sqrt{1-\frac{(0.99c)^2}{c^2}}}$$

$$= \frac{2 \times 10^{-6}}{\sqrt{1-0.99^2}} = 14.17 \times 10^{-6} sec$$

Twin Paradox

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

When v= C, Δt = ∞. The time interval of events in a frame of reference moving with velocity of light appears to be infinity to an observer in a stationery frame. This means that the passage of time in a very fastly moving spaceship appears to have stopped. Hence the process of aging will be stopped. This will explain Twin paradox.

Consider two synchronised clocks. One is kept on earth. The other is taken in a fast moving space ship. If the fast moving clock is brought back to earth after a long time, we can see that the time elapsed in moving clock will be less than the time elapsed in clock on earth. This is because a moving clock runs slow.

Actually there is no difference between a physical clock and a biological clock. Accordingly, heart beats of a person can be taken as clock. i.e. when a person moves with high speed, his heart beat will be slower. If his age is counted with reference to his heart beat, his age will be growing slower. Here comes the twin paradox.

Consider two identical twins A and B. A goes in a high speed spaceship which travels with a speed comparable to that of light. B stays at home. When A returns earth he will be younger than B. In relativity situations are inter changeable. Hence we can assume that A is rest and B is moving with velocity -v. After return B will be found to be younger than A. Both of them cannot be younger at the same time. This is a puzzle and is known as twin paradox. This can be explained on the basis of time dilation.

Actually the frames of reference of these two are different. For the brother at home the frame of reference remains the same. For the other brother the frame of reference has been changed due to his journey.

Proper frame, proper length and proper time

- The inertial frame of reference in which observed body is at rest is called the proper frame of reference.

- The length of a rod as measured in the inertial frame in which it is at rest is called the proper length.

  In the relation $l= l_0 \sqrt{1-\frac{v^2}{c^2}}$, l_0 is called the proper length.

- Like wise the proper time interval is the time interval recorded by a clock attached to the observed body.
- The relation between the proper time interval Δt_0 and non proper time (Δt) is as follows.

  $$\Delta t= \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Transformation equation for the velocity (Relativity)

Consider two inertial frames S and S’. S is at rest and S’ is moving with constant velocity v w.r.t. S along the X-axis. Let particle has a velocity u in the frame S and u’ be the velocity of the particle relative to S’.

The Lorentz transformation equations for the position co-ordinates and time are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

From these equations, we can write

$$dx' = \frac{dx - vdt}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad dt' = \frac{dt - \frac{vdx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \frac{dx'}{dt'} = u' = \frac{dx - vdt}{dt - \frac{vdx}{c^2}} = \frac{\frac{dx}{dt} - v\frac{dt}{dt}}{\frac{dt}{dt} - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u - v}{1 - \frac{uv}{c^2}}$$

(since $\frac{dx}{dt}$=u is the velocity in frame S)

This is the Lorentz transformation equation for the velocity in the x-direction.

The inverse transformation equation is $u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$

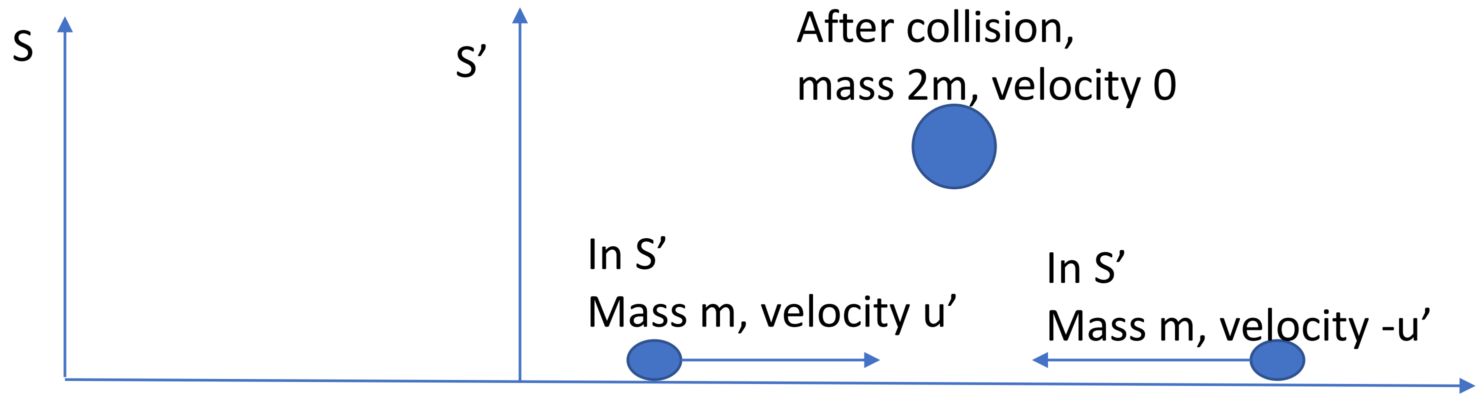
Problem: A photon is moving with the velocity of light c in an inertial frame S’, which also moves with a uniform velocity v. Show that the velocity of the photon remains the same.

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{c + v}{1 + \frac{cv}{c^2}}$$

$$= \frac{c + v}{1 + \frac{v}{c}} = \frac{c + v}{\frac{c + v}{c}} = c$$

Variation of mass with velocity

Conside two inertial frames S and S'. Let S be at rest and S' be moving with uniform velocity v along the positive direction of x-axis w.r.t. S.



Momentum before collision= momentum after collision

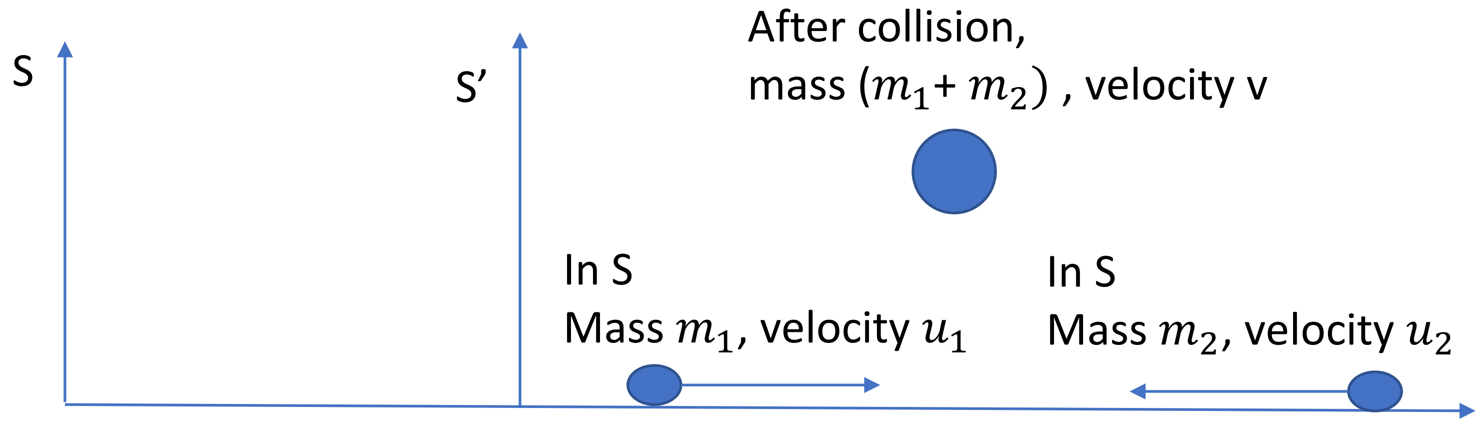
Momentum before collision = mu'-mu'=0

So, momentum after collision = 0

So velocity after collision = 0

Consider two identical bodies each of mass m moving in opposite directions ie, with velocities u' and –u' respectively along the x-axis w.r.t. S'. Let them collide. After collision the two bodies thick together and will be at rest in S', since the masses of the body are equal and momenta equal and oppositely directed.

If we observe the collision from the frame S let the masses of the two bodies appear as $m_1 and\ m_2\ and\ their\ velocities\ u_1 and u_2$ $before\ collision.$



Momentum before collision= momentum after collision

Momentum before collision $= m_1\ u_1 + m_2\ u_2$

momentum after collision = $(m_1 + m_2)$v

At the instant of collision these bodies come to rest w.r.t. S’ but move with a velocity v relative to S.

Applying the law of conservation of momentum in the frame S,

$m_1\ u_1 + m_2\ u_2 = (m_1 + m_2)$v-----------------------------------(1)

Applying the inverse velocity transformation,

$$u_1 = \frac{u_1{}' + v}{1 + \frac{u_1{}' v}{c^2}} = \frac{u' + v}{1 + \frac{u' v}{c^2}} \text{ and } u_2 = \frac{u_2{}' + v}{1 + \frac{u_2{}' v}{c^2}} = \frac{-u' + v}{1 - \frac{u' v}{c^2}}$$

Substitute for u_1 and u_2 in equation (1)

$$m_1 \left[\frac{u' + v}{1 + \frac{u' v}{c^2}}\right] + m_2 \left[\frac{-u' + v}{1 - \frac{u' v}{c^2}}\right] = (m_1 + m_2)\mathrm{v}$$

$$m_1 \left[\frac{u' + v}{1 + \frac{u' v}{c^2}} - v\right] + m_2 \left[\frac{-u' + v}{1 - \frac{u' v}{c^2}} - v\right] = 0$$

$$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}} - v\right] = -m_2\left[\frac{-u'+v}{1-\frac{u'v}{c^2}} - v\right]$$

$$\frac{m_1}{m_2} = -\frac{\left[\frac{-u'+v}{1-\frac{u'v}{c^2}} - v\right]}{\left[\frac{u'+v}{1+\frac{u'v}{c^2}} - v\right]} = -\frac{\left[\frac{-u'+v-v\left(1-\frac{u'v}{c^2}\right)}{1-\frac{u'v}{c^2}}\right]}{\left[\frac{u'+v-v\left(1+\frac{u'v}{c^2}\right)}{1+\frac{u'v}{c^2}}\right]} = \frac{\left[\frac{u'-v+v\left(1-\frac{u'v}{c^2}\right)}{1-\frac{u'v}{c^2}}\right]}{\left[\frac{u'+v-v\left(1+\frac{u'v}{c^2}\right)}{1+\frac{u'v}{c^2}}\right]}$$

$$= \left[\frac{u' - v + v\left(1 - \frac{u'v}{c^2}\right)}{1 - \frac{u'v}{c^2}}\right] \text{x} \left[\frac{1 + \frac{u'v}{c^2}}{u' + v - v\left(1 + \frac{u'v}{c^2}\right)}\right]$$

$$= \left[\frac{u' - v + v - \frac{u'v^2}{c^2}}{1 - \frac{u'v}{c^2}}\right] \text{x} \left[\frac{1 + \frac{u'v}{c^2}}{u' + v - v - \frac{u'v^2}{c^2}}\right] = \frac{1 + \frac{u'v}{c^2}}{1 - \frac{u'v}{c^2}}$$

From the equation $u_1 = \dfrac{u'+v}{1+\frac{u'v}{c^2}}$,

$$u_1^2 = \left(\frac{u'+v}{1+\frac{u'v}{c^2}}\right)^2 = \frac{u'^2+2u'v+v^2}{\left(1+\frac{u'v}{c^2}\right)^2} \Rightarrow \frac{u_1^2}{c^2} = \frac{\frac{u'^2}{c^2}+\frac{2u'v}{c^2}+\frac{v^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$\Rightarrow 1-\frac{u_1^2}{c^2} = 1-\frac{\frac{u'^2}{c^2}+\frac{2u'v}{c^2}+\frac{v^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2} = \frac{\left(1+\frac{u'v}{c^2}\right)^2-\left(\frac{u'^2}{c^2}+\frac{2u'v}{c^2}+\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$= \frac{1+\frac{2u'v}{c^2}+\frac{u'^2v^2}{c^4}-\frac{u'^2}{c^2}-\frac{2u'v}{c^2}-\frac{v^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2} = \frac{1-\frac{v^2}{c^2}+\frac{u'^2v^2}{c^4}-\frac{u'^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$= \frac{1-\frac{v^2}{c^2}+\frac{u'^2v^2}{c^4}-\frac{u'^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2} = \frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u'^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$\Rightarrow 1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2} \Rightarrow \left(1+\frac{u'v}{c^2}\right)^2 = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}$$

Similarly from the equation, $u_2 = \frac{-u'+v}{1-\frac{u'v}{c^2}}$, we can derive the expression

$$\left(1-\frac{u'v}{c^2}\right)^2 = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}$$

$$\frac{m_1}{m_2} = \frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} = \sqrt{\frac{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}}{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}}} = \sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}}$$

When u_1=0,let $m_1 = m_0$= rest mass. Then

$$\frac{m_0}{m_2} = \sqrt{1-\frac{u_2^2}{c^2}} \Rightarrow m_2 = \frac{m_0}{\sqrt{1-\frac{u_2^2}{c^2}}}$$

When u_2=0, let m_2= m_0= rest mass.   Then

$$\frac{m_1}{m_0} = \frac{1}{\sqrt{1-\frac{u_1^2}{c^2}}} \Rightarrow m_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

ie, If a particle of rest mass m_0 is moving with a velocity v, then its mass $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

m_0 is called the proper mass or rest mass

m is called the relativistic mass.

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Case 1- When the velocity v<<c, $\frac{v^2}{c^2}$ can be neglected.

Then m= m_0. Mass remains the same.

Case 2: when v=c, $\frac{v^2}{c^2}$=1, then m=∞.

When the particle is moving with the velocity of light, its mass becomes infinite and it is impossible. That is, the particle cannot have a velocity equal to c.

Case 3: When the velocity of the particle is close to the velocity of light $\frac{v^2}{c^2}$<1,

1- $\frac{v^2}{c^2}$<1.

Hence m> m_0. Then the relativistic mass increases.

Mass energy relation

According to work-energy theorem, when work is done on a particle, its kinetic energy increases and hence the velocity. If we do more work then the velocity will increase accordingly. But in special theory of relativity, the maximum speed that a particle can attain is c. Beyond that, the speed cannot be increased. Hence if we do more work on a particle, one part of that work is used to increase the mass of the particle. Thus mass and energy are interchangeable.

The relation between mass and energy is given by Einstein and is the Einstein's mass energy relation.

According to work energy theorem, the kinetic energy of the particle is given by

$$K=\int_0^r \vec{F}.\overrightarrow{dr}$$

$\vec{F}$ is the effective force acting on the particle

$\overrightarrow{dr}$ is the displacement.

But by second law of motion $\vec{F}=\frac{d}{dt}(m\vec{v})$

$$\therefore K=\int_0^v \frac{d}{dt}(m\vec{v}).d\vec{r} \quad = \int_0^v d(m\vec{v}).\frac{d\vec{r}}{dt} \quad = \int_0^v d(m\vec{v}).\vec{v}$$

$\therefore K=\int_0^v d(m\vec{v}).\vec{v} = \int_0^v (\vec{v}dm + md\vec{v}).\vec{v} = \int_0^v (\vec{v}.\vec{v}dm + m\vec{v}.d\vec{v})$

$=\int_0^v (v^2 dm + mvdv)$

From the equation $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, we can write $m^2=\frac{m_0^2}{\left(1-\frac{v^2}{c^2}\right)}$

$\Rightarrow m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$

$\Rightarrow m^2\left(\frac{c^2-v^2}{c^2}\right) = m_0^2$

$\Rightarrow m^2(c^2 - v^2) = m_0^2 c^2$

$\Rightarrow m^2c^2 - m^2v^2 = m_0^2 c^2$

On differentiating this equation(m^2c^2- m^2v^2= $m_0^2\ c^2$), we get

$c^2 2mdm - v^2 2mdm - m^2 2vdv$=0

Divide the equation by 2m

$c^2 dm - v^2 dm - mvdv$=0

$v^2 dm + mvdv$= $c^2 dm$

∴K= $\int_0^v (v^2 dm + mvdv)$ = $\int_{m_0}^{m} c^2 dm$ = $c^2[m]_{m_0}^{m}$

= $c^2[\mathrm{m} - m_0]$ = $mc^2 - m_0\ c^2$

The rest energy of the particle= $m_0\ c^2$

∴ The total energy E= Kinetic energy + rest energy

$$=mc^2 - m_0\ c^2 + m_0\ c^2$$
$$= mc^2$$

$$E\ = mc^2$$

This is the famous mass energy relation.

Examples of mass energy conversion

- Pair annihilation-When an electron collides with its antiparticle positron, they annihilate producing γ-rays. In this case, mass is converted into energy. The energy associated with the γ-rays is equal to the total energy associated with the mass of the two particles.
- Similarly when a high energy radiation having energy greater than $2m_0\,c^2$(m_0 is the rest mass of electron or positron) passes near the nucleus, a pair of electron and positron is produced. This is called Pair Production. Pair production is found in cosmic ray showers. Here the energy is converted to mass.
- The energy obtained from nuclear reaction like fission and fusion is a consequence of change of mass into energy.

Relativistic energy and momentum

For a particle of mass m moving with velocity v and having rest mass m_0,

Energy $E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$ ------------------------------------(1)

and

momentum $P = mv = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$ ------------------------------------(2)

Squaring equation (1) and (2)

$$E^2 = \frac{m_0^2 c^4}{\left(1-\frac{v^2}{c^2}\right)}$$ ------------------------------------(3)

$$P^2 = \frac{m_0^2 v^2}{\left(1-\frac{v^2}{c^2}\right)}$$ ------------------------------------(4)

Multiply equation for by c^2. Then

$$P^2c^2 = \frac{m_0^2 v^2 c^2}{\left(1-\frac{v^2}{c^2}\right)}$$ ---------------------------(5)

(3)-(5) =>

$$E^2 - P^2c^2 = \frac{m_0^2 c^4}{\left(1-\frac{v^2}{c^2}\right)} - \frac{m_0^2 v^2 c^2}{\left(1-\frac{v^2}{c^2}\right)} = \frac{m_0^2 c^4 - m_0^2 v^2 c^2}{\left(1-\frac{v^2}{c^2}\right)} = \frac{m_0^2 c^4\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)} = m_0^2 c^4$$

$$\therefore E^2 = P^2c^2 + m_0^2 c^4$$

$$\therefore E = \sqrt{P^2c^2 + m_0^2 c^4}$$

This is the relativistic energy - momentum relation.

The velocity of a massless particle is c

For a photon m_0=0.

E= $\sqrt{P^2c^2 + m_0^2c^4}$ => E= $\sqrt{P^2c^2}$ =Pc =mvc

But E=m c^2

∴ m c^2= mvc

⇒v=c.

⇒That is the speed of a massless particle is c.